An Analytical Comparison of Definite Numerical Integration Methods: A Study on Efficiency and Accuracy

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مقارنة تحليلية لطرق التكامل العددي المحدد: دراسة حول الكفاءة والدقة

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Abstract:
This study reviews definite numerical integration methods and provides an analytical comparison of different techniques used to compute integrals in the fields of applied mathematics and engineering. The primary goal of the research is to assess the efficiency and accuracy of each method in calculating definite integrals, with a focus on performance effectiveness when dealing with complex functions. A range of common methods, such as the trapezoidal rule, Romberg method, and Simpson's rule, were analyzed. The paper presents the results of numerical experiments conducted on a set of practical examples, with a detailed discussion of the challenges each method faces and its suitability for various applications. In conclusion, recommendations are provided for selecting the most appropriate method based on the required accuracy level and available computational resource.

Keywords: Numerical methods, integration, Numerical integration, Trapezoidal rule, Simpson's1\3 rule, Romberg method**.**

الملخص:
تستعرض هذه الدراسة طرق التكامل العددي المحدد وتقدم مقارنة تحليلية بين الأساليب المختلفة المستخدمة لحساب التكاملات في مجالات الرياضيات التطبيقية والهندسية. الهدف الرئيسي من البحث هو تقييم كفاءة ودقة كل طريقة من هذه الطرق في حساب التكاملات المحددة، مع التركيز على فعالية الأداء في حالة دوال معقدة. تم تحليل مجموعة من الأساليب الشائعة مثل طريقة شبه المنحرف، وطريقة رومبرج، وطريقة سيمبسون. يعرض البحث نتائج التجارب العددية التي تم إجراؤها على مجموعة من الأمثلة العملية، مع مناقشة تفصيلية للتحديات التي قد تواجهها كل طريقة وملاءمتها لأغراض تطبيقية مختلفة. في الختام، يتم تقديم توصيات حول اختيار الطريقة الأنسب بناءً على مستوى الدقة المطلوب والموارد الحسابية المتاحة.

الكلمات المفتاحية: طرق عددية، تكامل، تكامل عددي، قاعدة شبه المنحرف، قاعدة سيمبسون3/1، طريقة رومبرج.

Introduction:

Integration is considered one of the fundamental pillars of calculus, with its primary role being the calculation of accumulated quantities such as areas, volumes, displacements, and other physical or mathematical variables that change continuously. Integration is classified into two main types**: indefinite integration**, also known as the antiderivative, which represents the inverse process of differentiation; and **definite integration**, which is computed by substituting the bounds of the interval into the antiderivative. Burden, RL, Fairs, JD in [1]

Geometrically, definite integration in the coordinate plane represents the area enclosed between the function’s curve and the x-axis. This enables its application in determining areas, as well as in broader uses such as calculating volumes and other quantities resulting from the successive summation of infinitesimally small elements. However, many integrals cannot be solved using traditional analytical methods. Therefore, alternative approaches known as **numerical integration methods** have been developed to approximate integral values in cases where analytical solutions are not feasible. Jonh H, Mathew in [4]

Numerical analysis is a branch of applied mathematics concerned with the development and application of numerical methods to obtain approximate solutions for mathematical problems that are difficult or impossible to solve analytically. Since these methods rely on approximation, the resulting solutions are inherently approximate, which leads to the emergence of what is known as numerical error. It is essential to study and analyze this error to assess the accuracy and acceptability of the approximate solution, as the difference between the exact and approximate solutions serves as an indicator of the quality of the numerical method employed. Gerry Sozio in [2]

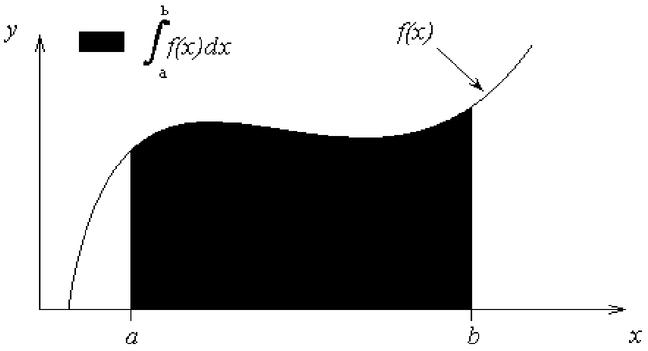
Numerical Integration is a branch of numerical analysis concerned with approximating the value of a definite integral for a given function when an analytical (explicit) solution is not feasible. This type of integration is used in cases where the function cannot be integrated symbolically or when it is given only through discrete numerical data. Gordon KS in [3]

This is accomplished by dividing the interval over which the integration is to be performed into small subintervals and estimating the area under the curve using numerical methods such as the **Trapezoidal Rule, Simpson's Rule,** or **Gaussian Quadrature.** Wu and Smolinski, in [9].
This paper aims to compare the Trapezoidal Rule and Simpson's Rule, both of which belong to the closed Newton-Cotes formulas, in addition to the Romberg integration method, which is derived from the Trapezoidal Rule. The comparison is conducted by applying these methods to a variety of definite integrals, with a focus on analyzing the results and examining the factors that influence the accuracy of the numerical outcomes.

Definition:

The process of determining the area under a curve represented by a function on a graph is known as definite integration. In this process**:**

$I = \int_a^b f(x)dx$, b> a . (1)



- **f(x)** represents the integrand (the function to be integrated).
- **a** is the **lower limit of integration**.
- **b** is the **upper limit of integration**.

Figure 1: *the Definite Integral*.

The result of the definite integral gives the total accumulated value (such as area) of the function **f(x)** over the interval [a, b]

Definition:

In numerical analysis, error is defined as the difference between the exact (true) value and the approximate value obtained using numerical methods. The sources of error can be attributed to several factors, including: inaccuracies in the input data, approximation inherent in numerical methods, truncation errors, computer-related errors (such as floating-point representation), and human error.[8]

Errors are typically quantified using three main approaches:

1. **Absolute Error:** The absolute difference between the true value and the approximate value.
2. **Relative Error:** The ratio of the absolute error to the exact value.
3. **Percentage Error:** The relative error multiplied by 100 to express it as a percentage.

Trapezoidal Rule Method:

The **Trapezoidal Rule** is a widely used numerical method for approximating the value of a definite integral. It works by approximating the curve of the function over the interval [a, b] with a straight line connecting the endpoints, and then computing the area under this line as an estimate of the area under the curve.**[7]**

The basic formula for the Trapezoidal Rule to approximate the integral of a function f(x) over the interval [a, b] is:

$I = \int_a^b f(x)dx$

In the context of numerical integration, the interval [a, b] is divided into n subintervals of equal length. The length of each subinterval, commonly referred to as the step size h, is calculated using the following formula:

$$h = \frac{b - a}{n}$$

Figure 2 illustrates how this partitioning is carried out, where the area under the curve is divided into n segments. Each segment resembles the shape of a trapezoid, and its area can be calculated using the trapezoidal rule.

$$A_1 = \frac{1}{2}(y_1 + y_2).h$$

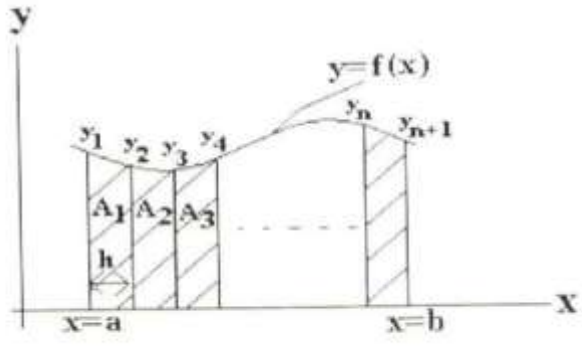


Figure 2: *Partition of the Definite Integral into Trapezoidal Segments Using the Trapezoidal Rule*.

To estimate the total area enclosed under the curve and between the two lines $x = a, x = b$, we sum the areas of all the individual segments A_1 , A_2 ,……., A_n as follows.

$$\int_a^b f(x)dx = Total\ Area = A_1{+}A_2{+}A_3{+}A_4 + \cdots \ldots + A_n \qquad \ldots (2)$$

$$A_1 = \frac{h}{2}(y_1 + y_2), A_2 = \frac{h}{2}(y_2 + y_3), A_3 = \frac{h}{2}(y_3 + y_4), \ldots., A_n = \frac{h}{2}(y_n + y_{n+1}) \ ..(3)$$

$$\int_a^b f(x)dx = \frac{h}{2}(y_1 + y_2) + \frac{h}{2}(y_2 + y_3) + \frac{h}{2}(y_3 + y_4) + \cdots + \frac{h}{2}(y_n + y_{n+1}) \quad \ldots(4)$$

$$\int_a^b f(x)\ dx = h(\frac{1}{2}y_1 + y_2 + y_3 + \cdots + y_n + \frac{1}{2}\ y_{n+1}) \qquad \ldots..(5)$$

The accuracy of the integral value depends on the number of segments used to divide the area under the curve. As the number of segments n increases, the accuracy of the integral improves. Therefore, n can be doubled after each integration step several times until the difference between the most recent result and the previous one becomes very small.[3]

Simpson's Rule Method:

Simpson’s Rule is a numerical method used to approximate the definite integral of a function. It is more accurate than the rectangle and trapezoidal methods, especially when the function is smooth.

To approximate the integral of a function f(x) over the interval [a, b] using the **Simple Simpson’s Rule** (where the interval is divided into 2 equal parts):[7]

$$\int_a^b f(x)\, dx = \frac{b-a}{6}\left[f(a) + 2f(\frac{a+b}{2}) + f(b)\right] \quad \ldots\ldots(6)$$

Notes:

- The number n must be even when using the composite form of Simpson’s Rule.
- Simpson’s Rule gives very accurate results for functions of degree three or lower.
- It is commonly used in numerical analysis and scientific computing to estimate integrals that are difficult to solve analytically.[1]

Simpson's 1/3 Rule **Method**

Simpson’s 1/3 Rule is the most common form of Simpson’s Rule. It is used to approximate definite integrals by fitting **parabolic segments** (instead of rectangles or trapezoids) under the curve.[8]

To approximate the integral:

$$\int_a^b f(x)dx$$

We divide the interval [a, b] into an even number of subintervals n, and comput

$$h = \frac{b - a}{n}$$

The simplest derivation of Simpson's 1/3 rule is done using Newton's forward difference formula.

$$f(x) = y_0 + \alpha.\Delta y_0 + \frac{\alpha(\alpha-1)}{2!}\Delta^2 y_0 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Delta^3 y_0 + \cdots \qquad ......(7)$$

Where $\alpha = \frac{x - x_0}{h}$, h is the step size and it is constant,

$$\Delta y_0 = y_1 - y_0$$

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\int_{x_0}^{x_2} f(x)dx = \int_{x_0}^{x_2} \left[y_0 + \alpha.\Delta y_0 + \frac{\alpha(\alpha-1)}{2!}\Delta^2 y_0\right] dx \qquad ......(8)$$

Thus, after integration and simplification, I obtain:

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] \qquad ....(9)$$

$$\int_{x_0}^{x_4} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4] \qquad ......(10)$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4] + \cdots + \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

$$I_{1/3} = \frac{h}{3}[y_0 + 4\sum_{i=odd}^{n-1} y_i + 2\sum_{i=even}^{n-2} y_i + y_n] \qquad ....(11)$$

Simpson's 3/8 Rule **Method**

If the number of subintervals is odd, it is preferable to use Simpson's 3/8 rule. In this method, the function f(x) is approximated using a cubic (third-degree) polynomial. The area under the curve over three subintervals can be calculated similarly to the derivation of Simpson's 1/3 rule, by applying Newton’s forward difference formula up to the fourth difference and then integrating from x_0 to x_3.[7]

$$\int_{x_0}^{x_2} f(x)dx = \int_{x_0}^{x_2} \left[y_0 + \alpha.\Delta y_0 + \frac{\alpha(\alpha-1)}{2!}\Delta^2 y_0 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Delta^3 y_0\right] dx. (12)$$

$$\Delta y_0 = y_1 - y_0$$

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

After performing the integration and simplification, I obtain the following result.

$$I_{3/8} = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] \qquad ...(13)$$

Example:

Evaluate the definite integral $\int_0^1 x^4\, dx$

using the following numerical methods:

1. The Trapezoidal Rule
2. Simpson’s 1/3 Rule
3. Simpson’s 3/8 Rule

Divide the interval [0, 1] into n= 6 **equal subintervals, and compare the numerical results with the exact value. Calculate the relative error for each method.**

The exact value of integral is**:**

$$\int_0^1 x^4\, dx = \frac{x^5}{5} = \frac{1}{5} = 0.2$$

$$h = \frac{b\ - a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

$y = x^4\,;\, x_1 = x_0 + h$

$x_2 = x_0 + 2h$

x	0	1/6	2/6	3/6	4/6	5/6	1
$y = x^4$	0	0.00077	0.01234	0.06251	0.1975	0.482253	1

Trapezoidal Rule Method**:**

$$I = h\left[\frac{1}{2}y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + \frac{1}{2}y_6\right]$$

$$I = \frac{1}{6}\left[\frac{1}{2}(0) + 0.00077 + 0.01234 + 0.06251 + 0.1975 + 0.482253 + \frac{1}{2}(1)\right]$$

$I = 0.20922$

Absolute Error= $|Exact - Approximate|$ =|0.2 − 0.20922 | =0.00922

Relative Error $= \frac{Exact - Approximate}{Exact} = \frac{0.00922}{0.2} = 0.0461$

Simpson's 1/3 Rule Method

$$I_{1/3} = \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6]$$

$$I_{1/3} = \frac{1/6}{3}[0 + 4(0.00077) + 2(0.01234) + 4(0.06251) + 2(0.1975) + 4(0.482253) + 1]$$

$I_{1/3} = 0.20010$

Absolute Error= $|Exact - Approximate|$ =|0.2 − 0.20010 | =0.0001

Relative Error $= \frac{Exact - Approximate}{Exact} = \frac{0.0001}{0.2} = 0.0005$

Simpson's 3/8 Rule Method

$$I_{3/8} = \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3)+(y_3 + 3y_4+3y_5 + y_6)]$$

$$I_{3/8} = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + 2y_3+3y_4+3y_5 + y_6]$$

$$I_{3/8} = \frac{1}{16}[0 + 3(0.00077) + 3(0.01234) + 2(0.06251) + 3(0.1975) + 3(0.482253) + 1]$$

$$I_{3/8} = 0.20022$$

Absolute Error= $|Exact - Approximate|$ =$|0.2 - 0.20022|$ =0.00022

Relative Error = $\frac{Exact - Approximate}{Exact} = \frac{0.00022}{0.2} = 0.0011$

Analysis of Results:

From the results, it is evident that:

Simpson's 1/3 Rule provided a highly accurate approximation of the integral, with a very small relative error. This is due to the fact that the rule approximates the function using quadratic (parabolic) segments, making it particularly effective for smooth functions.

Simpson's 3/8 Rule also yielded a high degree of accuracy, though slightly less precise than Simpson's 1/3 in this case, despite n=6n = 6n=6 being an exact multiple of 3. This slight difference may be attributed to the nature of the function, as cubic interpolation (used in Simpson's 3/8 Rule) does not always outperform quadratic interpolation, especially with a relatively small number of intervals.

The **Trapezoidal Rule** produced a reasonable approximation but was the least accurate among the tested methods. This outcome is expected, as the trapezoidal rule relies on linear interpolation, which cannot fully capture the curvature of smooth nonlinear functions.

Romberg Integration Method

Romberg integration is a numerical technique designed to approximate definite integrals with high accuracy by systematically refining estimates obtained from the trapezoidal rule. The method leverages Richardson extrapolation to accelerate convergence and reduce the error associated with numerical approximation. It begins with the application of the trapezoidal rule over successively halved subintervals, producing increasingly accurate estimates of the integral. These estimates are then organized into a Romberg table, where each successive column refines the previous one using the extrapolation formula:**[5]**

$$R_{2i}^{j} = \frac{4^j\, R_{2i}^{j-1} - R_{i}^{j-1}}{4^j - 1}, \qquad \ldots (14)$$

$$i = 1,2,4,8,16$$

$$j = 1,2,3, \ldots.$$

R_1					
R_2	R_2^1				
R_4	R_4^1	R_4^2			
R_8	R_8^1	R_8^2	R_8^3		
R_{16}	R_{16}^1	R_{16}^2	R_{16}^3	R_{16}^4	
R_{32}	R_{32}^1	R_{32}^2	R_{32}^3	R_{32}^4	R_{32}^5

R_{2i}^{j} represents the Romberg estimate at the i-th row and j-th column of the table. The method is particularly effective when the integrand is smooth and continuous, often yielding highly accurate results with relatively few function evaluations. As such, Romberg integration is widely used in scientific computing and engineering applications where precision is critical.[1]

Example:

1_Evaluate the definite integral $\int_0^4 x\ln x\, dx$

using the following numerical methods:

1. The Trapezoidal Rule
2. Simpson's 1/3 Rule
3. Simpson's 3/8 Rule
4. Romberg method, at $i = 1,2$

Divide the interval [0, 4] into n= 4 equal subintervals, and compare the numerical results with the exact value. Calculate the relative error for each method.

The exact value of integral is**:**

$$\int_0^4 x\ln(x)\, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} = [8\ln 4 - 4] = 7.09035$$

$$h = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

$$y = x\ln x\ ;\ x_1 = x_0 + h\ ,\ x_2 = x_0 + 2h, \dots, x_n = x_0 + nh$$

X	0	1	2	3	4
$y = x\ln x$	0	0	1.38694	3.29583	5.54517

Trapezoidal Rule Method**:**

$$I = h\left[\frac{1}{2}y_0 + y_1 + y_2 + y_3 + \frac{1}{2}y_4\right]$$

$$I = 7.45535$$

Absolute Error= $|Exact - Approximate|$ =$|7.09035 - 7.45535|$ =0.365

Relative Error $= \frac{Exact - Approximate}{Exact} = \frac{0.365}{7.09035} = 0.05147$

Simpson's 1/3 Rule Method

$$I_{1/3} = \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$I_{1/3} = 7.16745$$

Absolute Error= $|Exact - Approximate|$ =$|7.09035 - 7.16745|$ =0.0771

Relative Error $= \frac{Exact - Approximate}{Exact} = \frac{0.0771}{7.09035} = 0.01087$

Simpson's 3/8 Rule Method

$$I_{3/8} = \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + 2y_3 + y_4]$$

$$I_{3/8} = 6.11618$$

Absolute Error= $|Exact - Approximate|$ =|7.09035 − 6.11618| =0.97417

Relative Error = $\frac{Exact-Approximate}{Exact} = \frac{0.97417}{7.09035} = 0.13739$

Romberg method, $at\ i = 1,2$

To find j , compare the values of i with the values of j .

$i = 1,2,4,8,16$

$j = 1,2,3,4,5$. i stops at 2 and from there the value of j is 2.

$$R_{2i}^{j} = \frac{4^{j}\, R_{2i}^{j-1} - R_{i}^{j-1}}{4^{j} - 1}$$

$$R_{4}^{2} = \frac{4^{2}\, R_{4}^{1} - R_{2}^{1}}{4^{2} - 1}$$

R_1

R_2 $\quad R_2^1$

R_4 $\quad R_4^1$ $\quad R_4^2$

To find R_1, I used the trapezoidal method.

$R_1 = \frac{1}{2}(y_0 + y_1).h$, $h = \frac{b-a}{n} = \frac{4-0}{1} = 4$

$R_1 = \frac{4}{2}(y_0 + y_1)$; $y = xln(x)$, $x_0 = 0,\ x_1 = h + x_0 = 4$

$$R_1 = \frac{4}{2}(0 + 5.545) = 11.090$$

$R_2 = \frac{h}{2}(y_0 + 2y_1 + y_2)$, $h = \frac{b-a}{n} = \frac{4-0}{2} = 2$

$$R_2 = \frac{2}{2}(0 + 1.386 + 5.545) = 8.317$$

$$R_4 = \frac{h}{2}(y_0 + 2(y_1 + y_2 + y_3) + y_4) = 7.455$$

When i=1 and j=1 in Romberg method:

$$R_2^1 = \frac{4\, R_2^0 - R_1^0}{4-1} = \frac{4(8.317) - 11.090}{3} = 7.393$$

When i=2 , j=1.

$$R_4^1 = \frac{4\, R_4^0 - R_2^0}{4-1} = 7.168$$

When i =2 , j=2.

$$R_4^2 = \frac{4^2\, R_4^1 - R_2^1}{4^2 - 1} = 7.153$$

The exact value of integral is:

$$\int_0^4 x\ln(x)\,dx = \frac{x^2}{2}lnx - \frac{x^2}{4} = [8ln4 - 4] = 7.090$$

Error= $|Exact - Approximate|$ =|7.090 − 7.153 | =0.063

Relative Error = $\frac{Exact-Approximate}{Exact} = \frac{0.063}{7.09} = 0.0088$

Analysis of Results:

From the results, it is evident that:

- **Simpson's 1/3 Rule** provided very high accuracy.
- **Romberg Integration** delivered the most accurate result, as expected, due to its use of successive refinement and Richardson extrapolation.
- The **Trapezoidal Rule** yielded a reasonable approximation, but with lower accuracy than the higher-order methods.
- **Simpson's 3/8 Rule** using n=4 was less accurate in this example, likely because n=4 is not a multiple of 3, which affects the optimal performance of this rule.

Conclusion:

Different numerical integration methods offer varying levels of accuracy depending on the nature of the function and the number of subintervals used. The results demonstrate that higher-order polynomial approximation methods, such as Simpson's rules and Romberg integration, are generally more efficient and accurate, especially when applied to smooth functions. Romberg stands out as the best choice when maximum precision is required, while Simpson's 1/3 Rule offers an excellent balance between accuracy and simplicity of implementation

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