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Axisymmetric Circular Plate Element Stiffness Matrix Derivation By Explicit Integration Method FEM

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اشتقاق مصفوفة صلابة عنصر اللوح الدائري المتماثل محوريًا باستخدام طريقة التكامل الصريح بطريقة العناصر المحددة

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Abstract:
Explicit integration of stiffness matrix elements is generally applied in derivation of simple element matrices because of the tedious manual calculation needed when complex governing equations or interpolation function of high orders are involved. This work at derivation of element stiffness matrix of elastic axisymmetric circular plate with uniform thickness by explicit integration method and applying it in solving several problems under different types of loading and boundary conditions. Comparison between to finite element solutions and those obtained from corresponding theoretical equations showed good agreement.

Keywords: axisymmetric circular plate, explicit integration, finite element method.

الملخص:
تم تطبيق التكامل الصريح لعناصر مصفوفة الصلابة بشكل عام في اشتقاق مصفوفات العناصر البسيطة بسبب الحساب اليدوي الممل المطلوب عندما يتعلق الأمر بمعادلات حاكمة معقدة أو دالة تتضمن حساب دالة ذات درجات عالية. يهدف هذا البحث إلى استخلاص مصفوفة صلابة العناصر للوحة دائرية مرنة متناظرة المحور ذات سماكة موحدة بطريقة التكامل الصريح وتطبيقها في حل عدة مسائل تحت أنواع مختلفة من ظروف التحميل والحدود. أظهرت المقارنة بين حلول العناصر المحدودة وتلك التي تم الحصول عليها من المعادلات النظرية المقابلة توافقًا جيدًا.

الكلمات المفتاحية: اللوح الدائري المتماثل المحوري، التكامل الصريح، طريقة العناصر المحدودة.

1. Introduction

Theoretical investigations of deformation and stresses in circular and annular plates were carried out with those related to general forms of plates and shells. So, theoretical basis of axially symmetric circular and annular plates were presented in details in most of the well-known books discussing theory of plates and shells[1, 2, 3]. Many authors in finite element method considered the axisymmetric circular plate problem as an extension or example of other one-dimensional elements. For example, bathe[4] referred to the finite element solution of the circular plate and shell problems as axisymmetric cases of the isoparametric beam solution by using a typical three-node element, while zienkiewicz and Taylor [5] included the case of the axisymmetric circular plate as an example of axisymmetric shells. Reddy et al [6] presented unified finite element model that contained the classical theories of Euler Bernoulli, Timoshenko and simplified Reddy third-order beam. The axisymmetric bending of circular plates was included in the word as an extension of beam elements. In this work, elastic axisymmetric circular plate element with uniform thickness was developed by using bending moment equations and cubic Hermit interpolation functions. The resulting equations were integrated analytically to form the element stiffness matrix which used the solution many cases. Results were compared to those obtained by application of theoretical equations presented in related literature.

2. Finite Element Formulation

Generally, the derivation of the stiffness matrix is based on the governing equation of the deflection in axisymmetric bending.

$$\nabla^4 w = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = \frac{p}{D} \quad (1)$$

For fourth order continuity, a cubic displacement function was selected

$$w(\bar{r}) = a_1\bar{r}^3 + a_2\bar{r}^2 + a_3\bar{r} + a_4 \quad (2)$$

Where the a' are constant coefficients and $\bar{r}$ is the local radial coordinate over the element as shown in figure 1. In global coordinates terms, it can be expressed by

$$\bar{r} = r - r_1 \quad (3)$$

Application of deflection and rotation boundary conditions in equation (2) at $\bar{r} = 0$ and $\bar{r} = L$ yields the values of the coefficients. Thus, the displacement function could be written as product of interpolation function and displacement vectors as following

$$w(\bar{r}) = [N]\{d\} \quad (4)$$

where

$$[N] = [\ N_1\ \ N_2\ \ N_3\ \ N_4\ ], \qquad and\ \{d\} = \begin{Bmatrix} w_1 \\ \emptyset_1 \\ w_2 \\ \emptyset_2 \end{Bmatrix}$$

Where w_1 and w_2 are the nodal deflections in z direction and $\emptyset_1$and $\emptyset_2$are the nodal rotations about the tangential direction to element, as shown in figure 1. The resulting Hermite cubic interpolation functions [7] can be written as following

$$N_1[\bar{r}] = \frac{1}{L^3}(2\bar{r}^3 - 3L\bar{r}^2 + L^3) \quad (5)$$

$$N_2[\bar{r}] = \frac{1}{L^3}(L\bar{r}^3 - 2L^2\bar{r}^2 + L^3\bar{r}) \quad (6)$$

$$N_3[\bar{r}] = \frac{1}{L^3}(-2\bar{r}^3 + 3L\bar{r}^2) \quad (7)$$

$$N_4[\bar{r}] = \frac{1}{L^3}(L\bar{r}^3 - L^2\bar{r}^2) \quad (8)$$

Where L is element length.

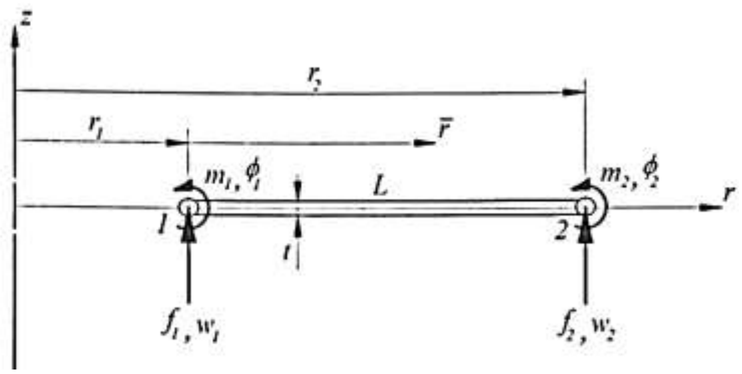

Figure 1: Axisymmetric circular plate element with local forces, moments and resulting deflections and rotations, respectively.

The strain / displacement relation of axisymmetric plate bending can be written as [8]

$$\varepsilon_r = -z\frac{d^2w}{dr^2} \tag{9}$$

$$\varepsilon_\theta = -z\frac{1}{r}\frac{dw}{dr} \tag{10}$$

Where ε_r is the radial strain and ε_θ is the tangential strain. By putting the two equations into the form of multiplication of the proper derivatives of the interpolation functions and the displacement vector, we get

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix} = -z\begin{bmatrix} \frac{d^2N_1}{dr^2} & \frac{d^2N_2}{dr^2} & \frac{d^2N_3}{dr^2} & \frac{d^2N_4}{dr^2} \\ \frac{1}{r}\frac{dN_1}{dr} & \frac{1}{r}\frac{dN_2}{dr} & \frac{1}{r}\frac{dN_3}{dr} & \frac{1}{r}\frac{dN_4}{dr} \end{bmatrix}\begin{Bmatrix} w_1 \\ \emptyset_1 \\ w_2 \\ \emptyset_2 \end{Bmatrix} \tag{11}$$

or

$$\{\varepsilon\} = [B]\{d\} \tag{12}$$

Hence, from equations (5) to (8) and (11), the strain-displacement matrix becomes

$$[B] = -\frac{z}{L^3}\begin{bmatrix} 12\bar{r} - 6L & 6L\bar{r} - 4L^2 & -12\bar{r} + 6L & 6L\bar{r} - 2L^2 \\ \frac{1}{r}(6\bar{r}^2 - 6L\bar{r}) & \frac{1}{r}(3L\bar{r}^2 - 4L^2\bar{r} + L^3) & \frac{1}{r}(-6\bar{r}^2 + 6L\bar{r}) & \frac{1}{r}(3L\bar{r}^2 - 2L^2\bar{r}) \end{bmatrix} \tag{13}$$

The corresponding stresses can be found from elastic relations as [3]

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = \frac{E}{1 - v^2}\begin{bmatrix} 1 & v \\ v & 1 \end{bmatrix}\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix} \tag{14}$$

or

$$\{\sigma\} = [D]\{\varepsilon\} \tag{15}$$

Where $[D]$ is the constitutive matrix in case of plate bending.

The total potential energy for axisymmetric plate bending is:

$$\pi_p = U + \Omega \tag{16}$$

Where U is the element strain energy:

$$U = \iiint_v^1 \frac{1}{2}\sigma\varepsilon dv \tag{17}$$

and Ω is the potential energy distributed load over element length and applied nodal concentrated forces and moments:

$$\Omega = -\iint_A^1 pwdA - \sum_{i=1}^{2} p_i w_i - \sum_{i=1}^{2} M_i \emptyset_i \tag{18}$$

By writing equations (17) and (18) in terms of axisymmetric plate quantities and limits, equations (16) becomes

$$\pi_p = \int_{r_1}^{r_2}\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_0^{2\pi} \frac{1}{2}\{\sigma\}^T\{\varepsilon\} r d\theta dz dr$$

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$$-\int_{r_1}^{r_2}\int_0^{2\pi} pw(r)rd\theta dr - 2\pi\sum_{i=1}^{2}(P_i w_i + M_i\phi_i)r_i \qquad (19)$$

Now, substitution equations (15),(12) and (4) into (19), performing simple integrations along circumference ($d\theta$) and differentiation with respect to displacement vector elements with equating to zero for minimum potential energy yields

$$2\pi\int_{r_1}^{r_2}\int_{-\frac{t}{2}}^{\frac{t}{2}} [B]^T[D][B][d]rdzdr - 2\pi p\int_{r_1}^{r_2} [N]^T rdr - 2\pi\{\bar{P}\} = 0 \qquad (20)$$

Where

$$\{\bar{P}\} = \begin{Bmatrix} P_1 w_1 \\ M_1\phi_1 \\ P_2 w_2 \\ M_2\phi_2 \end{Bmatrix}$$

The element stiffness matrix can be easily obtained from equation (20) and written as

$$[k] = 2\pi\int_{r_1}^{r_2}\int_{-\frac{t}{2}}^{\frac{t}{2}} [B]^T[D][B]rdzdr \qquad (21)$$

In order to complete the derivation, the strain-displacement matrix $[B]$ of equation (13) should be expressed in terms of global r-coordinates by using equation (3) to perform the integration of equation (21). To simplify matrix multiplication, the equations, strain and constitutive matrices can be rewritten as

$$[B] = -\frac{z}{L^3}[\bar{B}], \quad and \quad [D] = \frac{E}{1-v^2}[\bar{D}]$$

and by performing integration across element thickness, the stiffness matrix becomes in the following form

$$[k] = \frac{2\pi D_f}{L^6}\int_{r_1}^{r_2}[\bar{B}]^T[\bar{D}][\bar{B}]rdr \qquad (22)$$

Where D_f is the flexural rigidity of the plate:

$$D_f = \frac{Et^3}{12(1-v^2)} \qquad (23)$$

The final step in the stiffness matrix derivation is evaluation of equation (22). This can be performed by one of three methods [9]:

1. Numerical integration
2. Explicit multiplication of matrices then term-by-term integration
3. Evaluate $[B]$ at middle point instead of the entire element length instead of integration.

In this work, the explicit matrix multiplication is performed the elements of resulting 4×4 matrix are integrated manually from r_1 to r_2 . The resulting elements of stiffness matrix are as following

$$k_{11} = \frac{2\pi D}{L^6}\{9(5+4v)\Delta r^4 - 72(L+2r_1)(1+v)\Delta r^3$$

$$+36[L^2(1+v) + r_1(L+r_1)(5+6v)]\Delta r^2$$

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$$-72r_1(L+2r_1)(L+r_1)(1+v)\Delta r+[6r_1(L+r_1)]^2\Delta\ln r\} \qquad (24)$$

$$k_{12}=k_{21}=\frac{2\pi D}{L^6}\{\frac{9}{2}L(5+4v)\Delta r^4-6L(7L+12r_1)(1+v)\Delta r^3$$
$$+3L[r_1(7L+6r_1)(5+6v)+L^2(9+10v)]\Delta r^2$$
$$-6L(L+r_1)[L^2+3r_1(3L+4r_1)](1+v)\Delta r$$
$$+6Lr_1(L+3r_1)(L+r_1)^2\Delta\ln r\} \qquad (25)$$

$$k_{13}=k_{31}=-k_{11} \qquad (26)$$

$$k_{14}=k_{41}=\frac{2\pi D}{L^6}\{\frac{9}{2}L(5+4v)\Delta r^4-6L(5L+12r_1)(1+v)\Delta r^3$$
$$+3L[r_1(5L+6r_1)(5+6v)+4L^2(1+v)]\Delta r^2$$
$$-6Lr_1[4L^2+3r_1(5L+4r_1)](1+v)\Delta r$$
$$+6L{r_1}^2(2L+3r_1)(L+r_1)\Delta\ln r\} \qquad (27)$$

$$k_{22}=\frac{2\pi D}{L^6}\{\frac{9}{4}L^2(5+4v)\Delta r^4-12L^2(2L+3r_1)(1+v)\Delta r^3$$
$$+L^2[3r_1(4L+3r_1)(5+6v)+L^2(19+22v)]\Delta r^2$$
$$-4L^2(2L+3r_1)(L+3r_1)(L+r_1)(1+v)\Delta r$$
$$+L^2[L^4+8L^3r_1+22L^2{r_1}^2+24L{r_1}^3+9{r_1}^4]\Delta\ln r\} \qquad (28)$$

$$k_{23}=k_{32}=-k_{12} \qquad (29)$$

$$k_{24}=k_{42}=\frac{2\pi D}{L^6}\{\frac{9}{4}L^2(5+4v)\Delta r^4-18L^2(L+2r_1)(1+v)\Delta r^3$$
$$+\frac{1}{2}L^2[18r_1(L+r_1)(5+6v)+L^2(19+22v)]\Delta r^2$$
$$-2L^2(L+2r_1)[L^2+9r_1(L+r_1)](1+v)\Delta r$$
$$+L^2r_1(2L+3r_1)(L+3r_1)(L+r_1)\Delta\ln r\} \qquad (30)$$

$$k_{33}=k_{11} \qquad (31)$$

$$k_{34}=k_{43}=-k_{14} \qquad (32)$$

$$k_{44}=\frac{2\pi D}{L^6}\{\frac{9}{4}L^2(5+4v)\Delta r^4-12L^2(2L+3r_1)(1+v)\Delta r^3$$
$$+\frac{1}{2}L^2[6r_1(2L+3r_1)(5+6v)+8L^2(1+v)]\Delta r^2$$
$$-4L^2r_1(2L+3r_1)(L+3r_1)(1+v)\Delta r+[Lr_1(2L+3r_1)]^2\Delta\ln r\} \qquad (33)$$

Where $\Delta r^n=({r_2}^n-{r_1}^n)$ and $\Delta\ln r=lnr_2-lnr_1$.

Finally, to determine the forces applied on the element, the lode terms of equation (20) should be defined. While the concentrated loads act on nodes directly, the distributed load over the element should be interpolation to from nodal forces vector , f, by the integral .

$$\{f\} = 2\pi p \int_{r_1}^{r_2} [N]^T \, r dr \tag{34}$$

So, integration of the interpolation functions yields the following vector elements

$$f_1 = 2\pi p\{\frac{2}{5}\Delta r^5 - \frac{3}{4}(L + 2r_1)\Delta r^4 + 2r_1(L + r_1)\Delta r^3 + \frac{1}{2}(L^3 - 3Lr_1{}^2 - 2r_1{}^3)\Delta r^2\} \tag{35}$$

$$m_1 = 2\pi p\{\frac{1}{5}L\Delta r^5 - \frac{1}{4}L(2L + 3r_1)\Delta r^4 + \frac{1}{3}(L + 3r_1)(L + r_1)\Delta r^3 - \frac{1}{2}Lr_1(L + r_1)^2\Delta r^2\} \tag{36}$$

$$f_2 = 2\pi p\{-\frac{2}{5}\Delta r^5 + \frac{3}{4}(L + 2r_1)\Delta r^4 - 2r_1(L + r_1)\Delta r^3 + \frac{1}{2}r_1{}^2(3L - 2r_1)\Delta r^2\} \tag{37}$$

$$m_2 = 2\pi p\{\frac{1}{5}L\Delta r^5 - \frac{1}{4}L(L + 3r_1)\Delta r^4 + \frac{1}{3}Lr_1(2L + 3r_1)(L + r_1)\Delta r^3 - \frac{1}{2}Lr_1{}^2(L + r_1)^2\Delta r^2\} \tag{38}$$

Another approach in the derivation of the stiffness matrix is by using equations of radial and tangential bending moments in axisymmetric plate [1, 3] and the same results are obtained. In this work, a procedure-oriented C++ code, named *axiplate*, was programmed to utilized the derived stiffness matrix and nodal force vector equations. The global stiffness matrix and force vector were assembled using superposition method [9], and system was solved by a Gauss-elimination routine.

3. Results

To examine the validity to results obtained by the derived stiffness matrix, several cases were solved and results are compared with those obtained corresponding theoretical equations from reference [3]. The common data of the cases solved in this work was as following

Radius $1.0\ m$

Thickness $0.01\ m$

modulus of elasticity $200 \times 10^9\ N/m^2$

applied load $-10.0\ N\ or\ N/m$

poison's ratio 0.3

The applied load could be uniformly distributed load or concentrated force.

3.1. Uniformly distributed load

Figure (2) and (3) show the distribution of deflection and stresses along the radius of a circular plate with rigidly clamped edge.

Figure (4) and (5) show the solutions of a circular plate with simply supported edge.

3.2. Central concentrated force

Figure (6) and (7) show the solutions of a circular plate with clamped edge and figure (8) and (9) show those of simply supported plate.

3.3. Variation of uniformly distributed and concentrated loads over radial distance

The program was applied when loading position in varying from center to edge [3]. A 8-element model was used with the general date mentioned above. In figure (10.1), (10.2) and (11), the behavior of deflection at the center and radial stresses at the center and the edge of a clamped plate under distributed and concentrated loads, respectively. Figure (12) and (13) show the deflection and the radial stress at the plate center under distributed and concentrated loads when edge is simply supported.

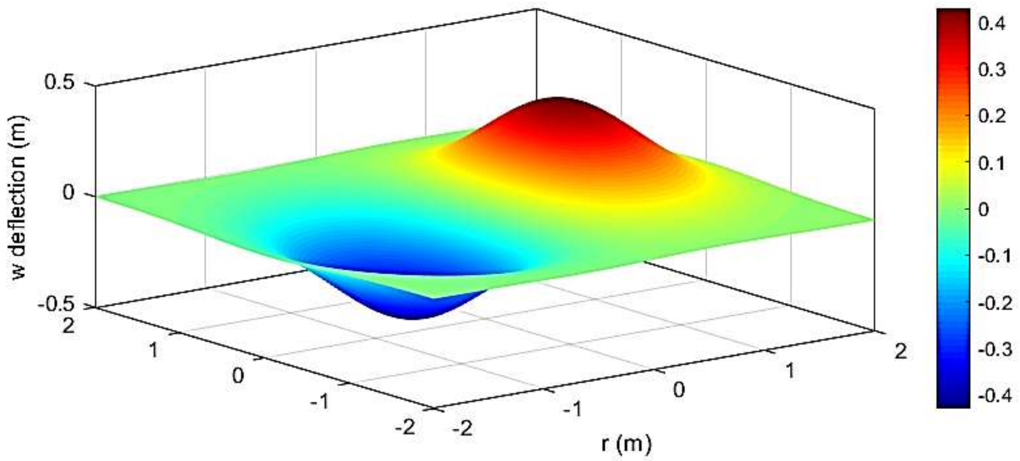

Figure 2: Deflections, w , along the radius of a circular plate with clamped edge under uniformly distributed load.

4. Discussion

Generally, a very good agreement between the finite element and the theoretical solution is seen and convergence was obtained with small number of elements. Distributed load cases converged more rapidly to exact solutions than those under a central concentrated force. This was due to the effect of application of the force on a point where $r = 0$ on the formation of finite element equations system. Theoretically, this is noticed at application of the force over a tiny area at center, and for this reason the equivalent radius was defined [3] to avoid zero radius substitution in logarithmic terms existing in the most axisymmetric plate equations.

At first glance, the explicit multiplication of matrices in equation (22) element-by-element then integration of resulting elements of the stiffness matrix might seem to the tedious and error prone process when performed manually, but three main advantages were obtained:(1) the accuracy of analytic integration in comparison to other methods, (2) the minimal programming effort and (3) the quicker run of the program. For more complicated 2- and 3- dimensional elements, commercial packages, such as MATLAB, supply symbolic math tools that can be utilized in matrix multiplication and integration.

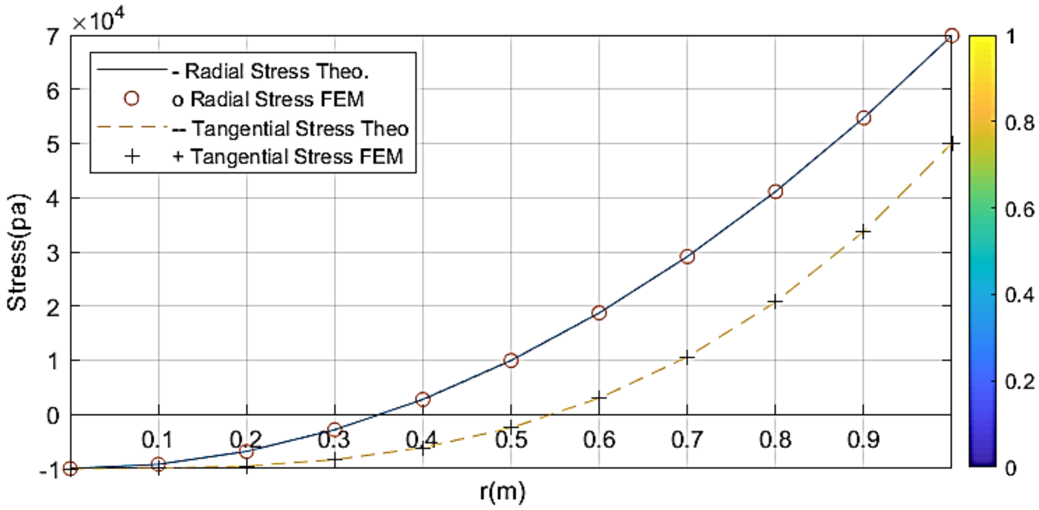

Figure 3: Stresses, σ_r and σ_θ, along the radius of a circular plate with clamped edge under uniformly distributed load.

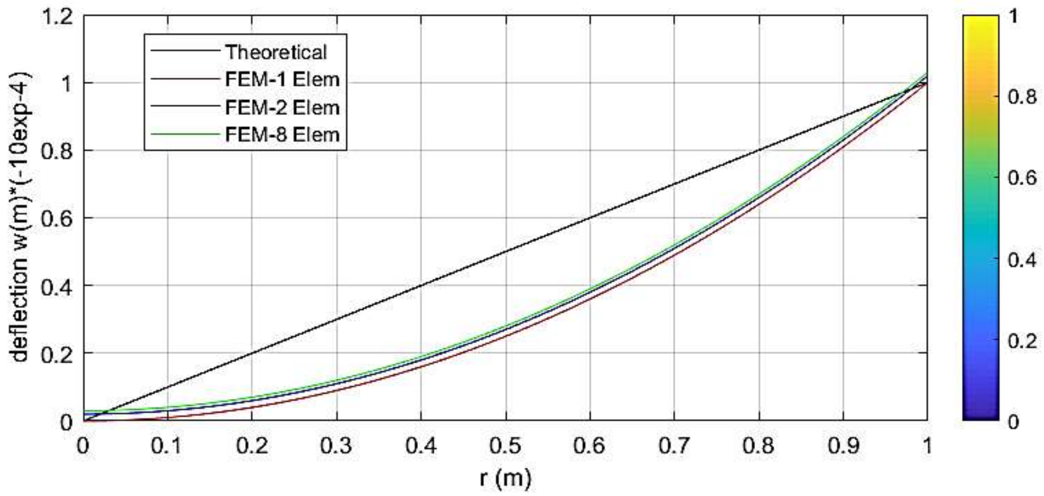

Figure 4: Deflections, w, along the radius of a circular plate with simply supported edge under uniformly distributed load.

5. Conclusions

The obtained stiffness matrix of axisymmetric circular plates yielded remarkable convergence to theoretical solution in the cases analyzed in this work. The simplicity of the computer code and speed of run are the main advantages of the used derivation method. The code can be executed without any modifications to annular plates of uniform thickness.

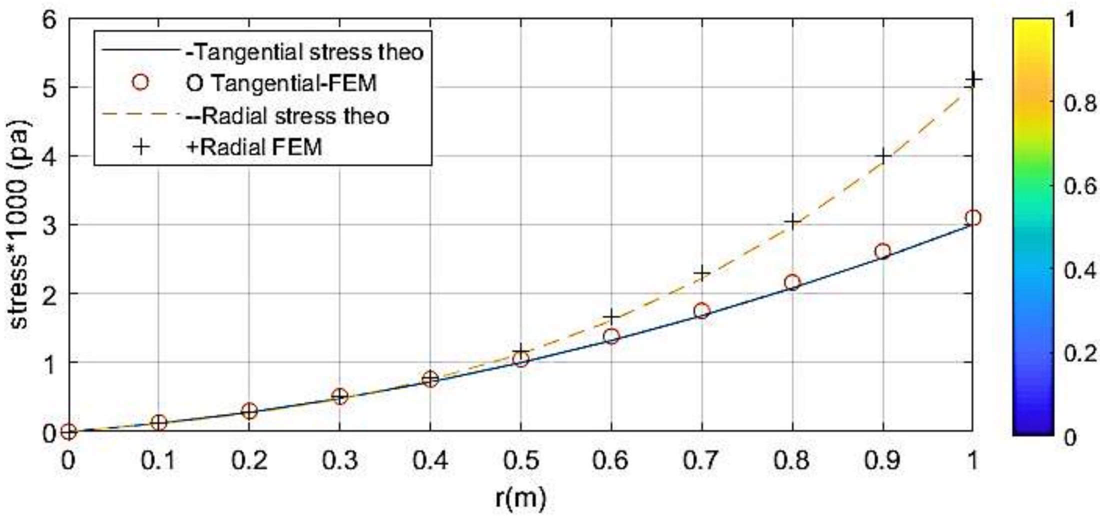

Figure 5: Stresses, σ_r and θ_r ,along the radius of a circular plate with simply supported edge under uniformly distributed load.

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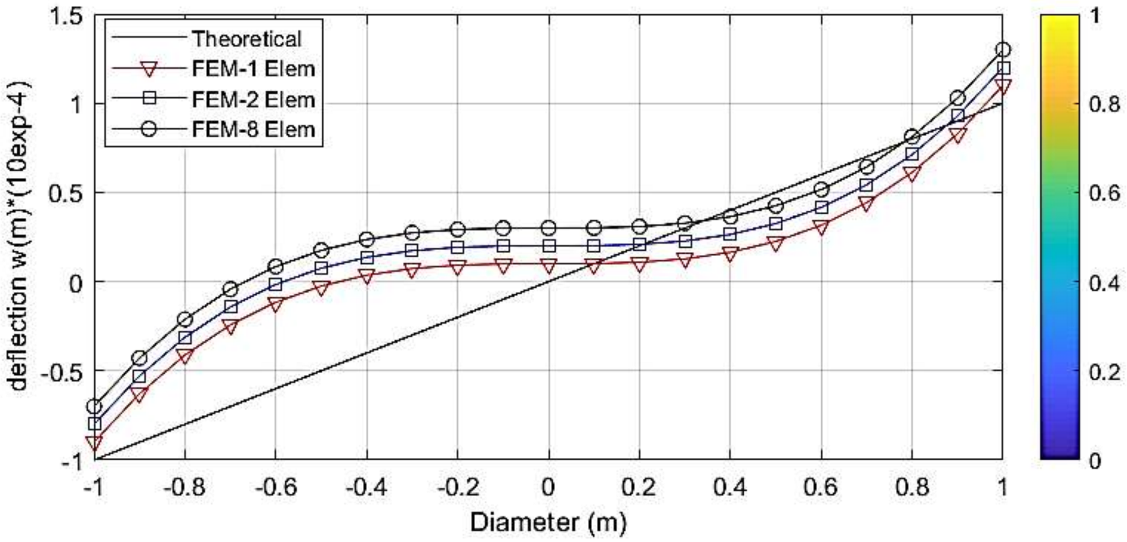

Figure 6: Deflections, w, along the radius of a circular plate with clamped edge under concentrated force at center.

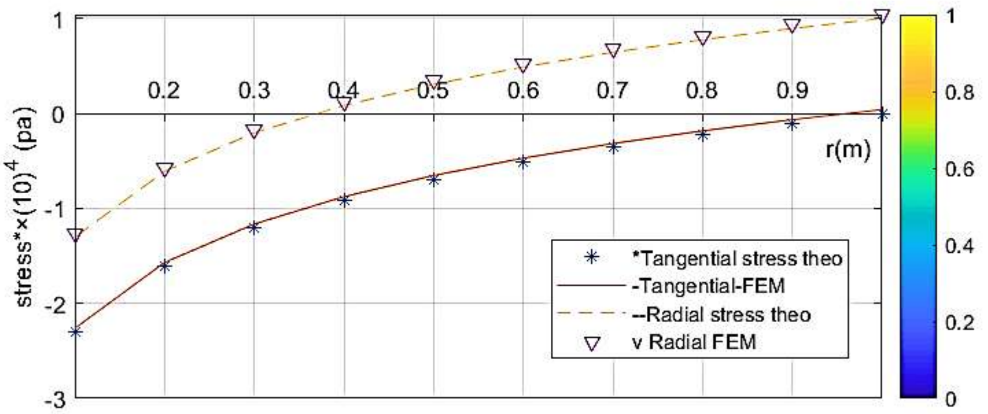

Figure 7: Stresses, σ_r and θ_r ,along the radius of a circular plate with clamped edge under concentrated force at center.

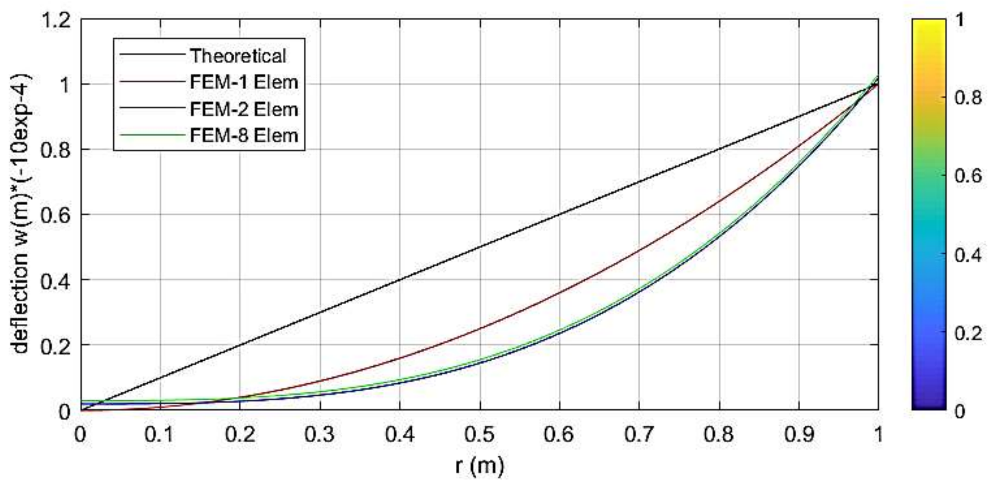

Figure 8: Deflections, w, along the radius of a circular plate with simply supported edge under concentrated force at center.

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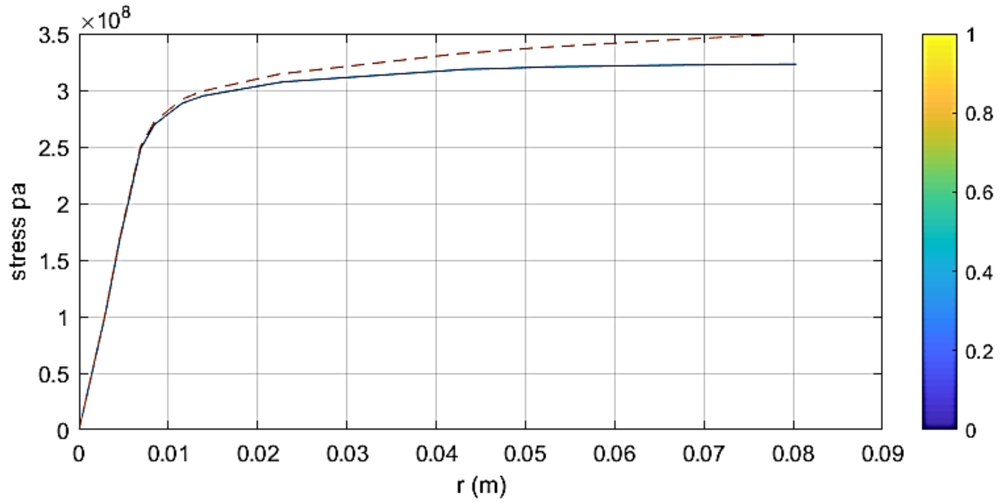

Figure 9: Stresses, σ_r and σ_θ ,along the radius of a circular plate with simply supported edge under concentrated force at center.

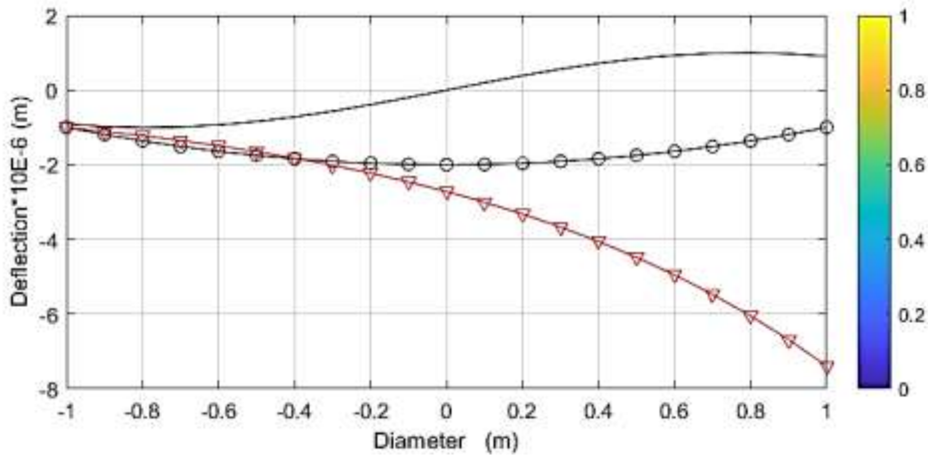

Figure 10.1: Center deflection w, at center along the radius of a circular plate with clamped plate under varying distributed load.

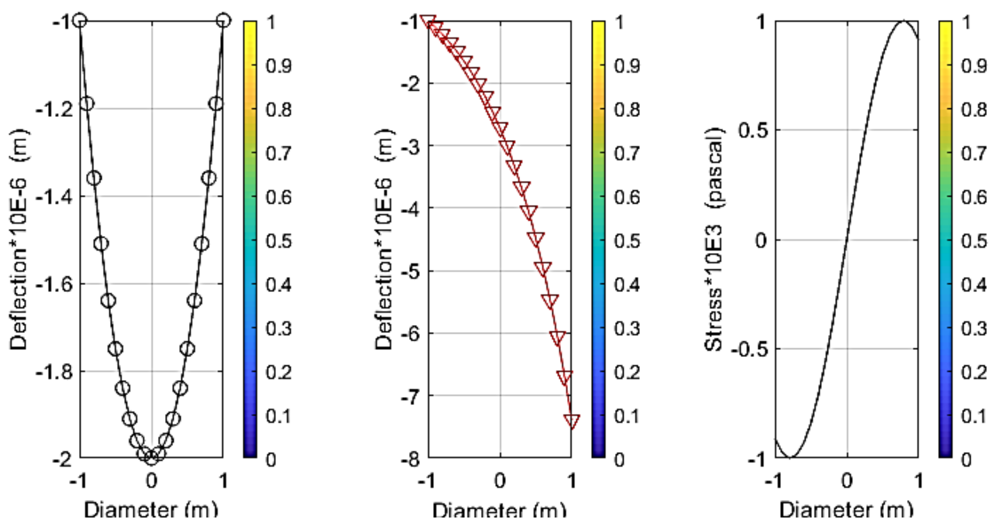

Figure 10.2: Center deflection w, at center along the radius and edge radial stresses, σ_r ,in a clamped plate under varying distributed load.

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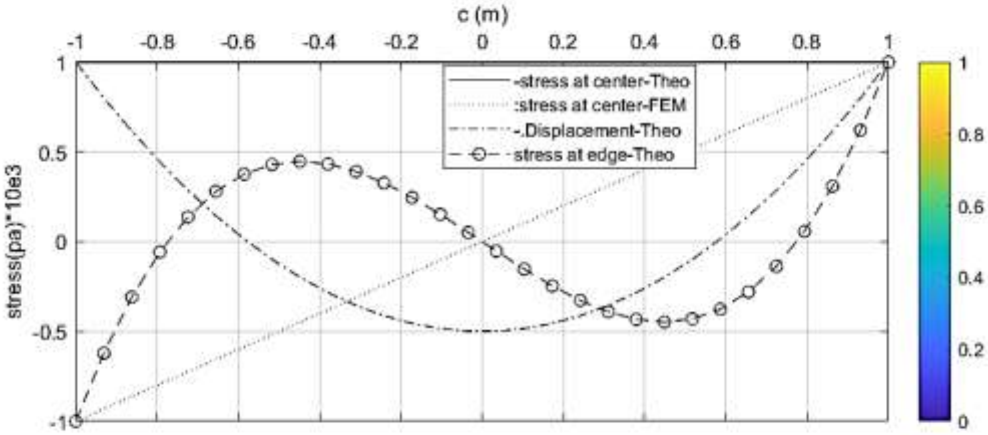

Figure 11: Radial stress σ_r along the radius of a circular plate with simply supported plate under varying distributed load.

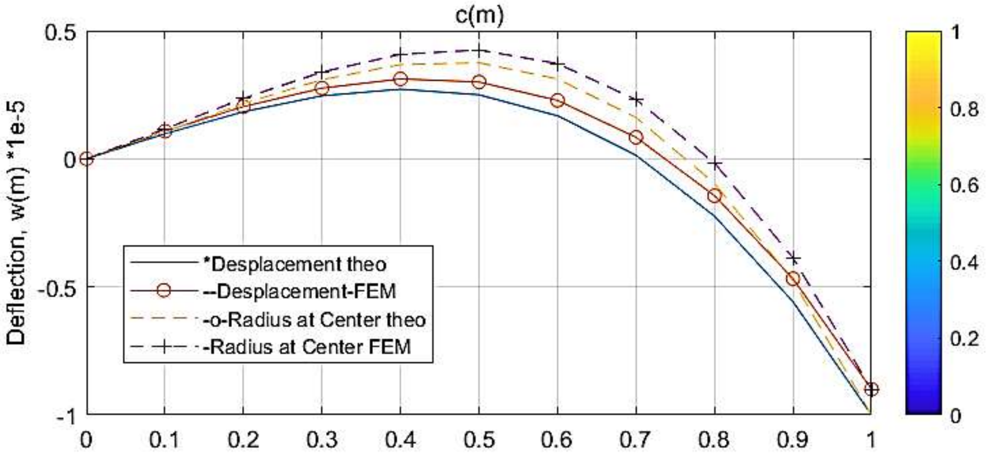

Figure 12: Center deflection w,and along the radius of a circular plate with simply supported plate under varying distributed load.

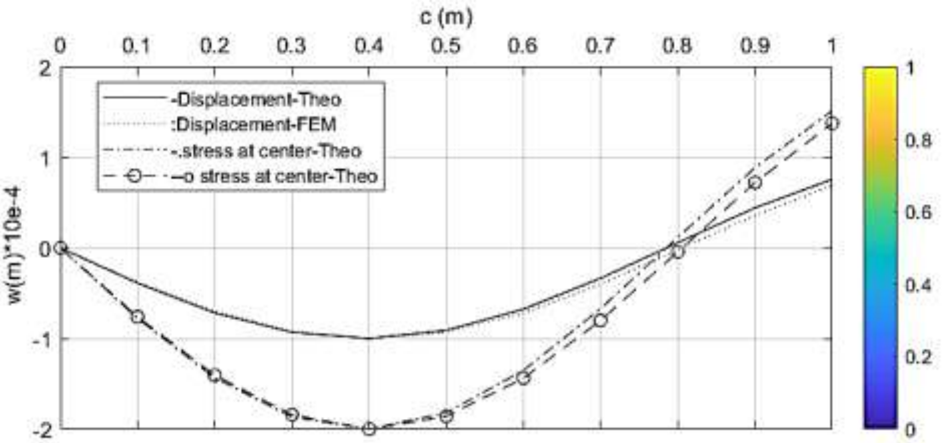

Figure 13: Center deflection w,and along the radius with a radial stress σ_r in a simply supported plate under varying distributed force.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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