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Some Properties and Bounds of Polygamma Function

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حدود عليا وسفلى جديدة لدالة جي- بتمان

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قسم الرياضيات، كلية التربية، جامعة بنغازي، بنغازي، ليبيا

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Abstract

This paper aims to study and present some properties related of the Polygamma functions. In particular, it is proved that for $x > 0$ and $n = 1, 2, \ldots$The function $\frac{\psi^{(n)}(x)}{\psi^{(n+1)}(x)} + \frac{x}{n}$, is increasing.

Also, it is provided the double inequality

$$0 < \frac{\psi^{(n)}(x)}{\psi^{(n+1)}(x)} + \frac{x}{n} < \frac{1}{2n}$$

Keywords: digamma function; bounds; Polygamma function.

الملخص

الهدف من هذه الدراسة هو تقديم خصائص جديدة لتركيبة من الدوال الخاصة من دوال بوليغاما وتتمثل هذه الخصائص في
تزايد دالة $\frac{\psi^{(n)}(x)}{\psi^{(n+1)}(x)} + \frac{x}{n}; x > 0 \ , \ n = 1, 2, \ldots$ على الفترة من صفر إلى ما لا نهاية

كما تم إثبات أن هذه التركيبات الخاصة مقيدة بحدود دنيا وعليا بالمتباينة

$$0 < \frac{\psi^{(n)}(x)}{\psi^{(n+1)}(x)} + \frac{x}{n} < \frac{1}{2n}$$

الكلمات المفتاحية: دالة ديجاما، الحدود ، دالة بولي جاما،

Introduction

The digamma function ψ(x) is the logarithmic derivative of gamma function Γ(x) and is defined by

$$\psi(x) = \frac{d}{dx}\log \Gamma(x) \, , x \neq 0, -1, -2, \ldots \qquad (3.1)$$

The digamma function satisfies the recurrence formulas

$$\frac{1}{x} = \psi(x + 1) - \psi(x), x \neq 0, -1, -2, \ldots \qquad (3.2)$$

It may also be defined as the limit

$$\psi(x) = \lim_{n\to\infty}\left(ln\, x - \sum_{k=1}^{n} \frac{1}{x + k} \right) \qquad (3.3)$$

For more details, refer to [1]-[2] and the references therein.

In [3], the asymptotic expansion is as follows

$$\psi(x) \sim ln\, x - \frac{1}{2x} - \frac{1}{12x^2} = ln\, n - \sum_{n=1}^{\infty} \frac{B_k}{kx^k} \quad x \to \infty \qquad (3.4)$$

Where B_n , $n = 0,1,2, \ldots$, are the Bernoulli numbers.

$$\psi\left(\frac{1+x}{2}\right) - \psi\left(\frac{x}{2}\right) \sim \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(2^{2n-} - 1)B_{2n}}{nx^{2n}}, \quad x \to \infty \ , \ x \neq 0, -1, -2, \qquad (3.5)$$

Recently in 2022 [5], Mahmoud, Talat, and Ahfaf introduced the following approximation

$$\psi\left(\frac{1+x}{2}\right)-\psi\left(\frac{x}{2}\right)=\frac{1}{x}+\frac{x^{-2}}{2(1+\sum_{i=1}^{n} b_i x^{-2i})}+O\left(\frac{1}{x^{2n+2}}\right), \qquad (3.6)$$

where $b_1 = \frac{1}{2}$, and

$$b_i = -2\left(\frac{(2^{2i+2}-1)B_{2i+2}}{i+1}+\sum_{v-1}^{i-1}\frac{(2^{2i-2v+2}-1)B_{2i+2v+2}}{i-v+1}b_v\right), \qquad i > 1.$$

The nth derivative of $\psi(x)$[1]

$$\psi(x)^{(n)} = \psi^{(n)}(x+1)-\frac{(-1)^n n!}{x^{n+1}}, n = 0,1,\ldots \qquad (3.7)$$

also, from [4] we have

$$\psi^{(n)}(x)=\frac{(-1)^n(n-1)!}{x^n}-\frac{(-1)^n(n)!\,2}{x^{n+1}}+O\left(x^{-(n+2)}\right), n = 1,2,\ldots \quad (3.8)$$

In [6] (pp. 10), and [7] (p. 33) The asymptotic representations of functions contributing to the polygamma functions are given in the following. We maintain contributions up to $O\left(\frac{1}{x^{19}}\right)$, to reach double precision accuracy in a fully analytic representation. Higher order terms can calculated if needed in more special numerical applications.

$$\psi^{(n)}(x)\sim(-1)^{n+1}\sum_{i=0}^{\infty}\left(\frac{(i+n-1)!}{i!}\right)\cdot\frac{B_i}{x^{i+n}}, \quad n \geq 1 \qquad (3.9)$$

$$\psi^{(n)}(x)\sim(-1)^{n-1}\left[\frac{(n-1)!}{x^n}+\frac{(n)!}{2x^{n+1}}+\sum_{r=1}^{\infty}\frac{(n+2r-1)!\,B_{2r}}{(2r)!\,x^{n+2r}}\right], \qquad x\to\infty \quad (3.10)$$

$$\psi(x)\sim \ln x-\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\frac{1}{252x^6}+\frac{1}{240x^8}-\frac{1}{123x^{10}}+\frac{691}{32760x^{12}}-\frac{1}{12x^{14}}+\cdots+\frac{174611}{6600x^{20}} + O\left(\frac{1}{x^{22}}\right)$$

$$\psi^{(1)}(x)\sim\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}+\frac{1}{42x^7}-\frac{1}{30x^9}+\frac{5}{66x^{11}}-\frac{691}{2730x^{13}}+\cdots+\frac{174611}{330x^{21}}+O\left(\frac{1}{x^{23}}\right)$$

$$\psi^{(2)}(x)\sim-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{2x^4}+\frac{1}{6x^6}-\frac{1}{6x^8}+\frac{3}{10x^{10}}-\frac{5}{6x^{12}}+\frac{691}{210x^{14}}+\cdots-\frac{43867}{42x^{20}}+O\left(\frac{1}{x^{22}}\right)$$

$$\psi^{(3)}(x)\sim\frac{2}{x^3}+\frac{3}{x^4}+\frac{2}{x^5}-\frac{1}{x^7}+\frac{4}{3x^9}-\frac{3}{x^{11}}-\frac{10}{x^{13}}-\frac{691}{15x^{15}}\ldots+\frac{438670}{21x^{21}}+O\left(\frac{1}{x^{23}}\right)$$

$$\psi^{(4)}(x)\sim-\frac{1}{6x^4}-\frac{12}{x^5}-\frac{10}{x^6}+\frac{7}{x^8}-\frac{12}{x^{10}}+\frac{33}{x^{12}}-\frac{130}{x^{14}}+\frac{691}{x^{16}}+\cdots-\frac{206169}{5x^{20}}+O\left(\frac{1}{x^{22}}\right)$$

Also, we have [4] the asymptotic expansions:

$$\psi^{(1)}(x)=\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}+\frac{\theta_1}{30x^5}, \qquad 0 < \theta_1 < 1$$

$$\psi^{(2)}(x)=-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{2x^4}+\frac{1}{6x^6}-\frac{\theta_2}{6x^8}, \qquad 0 < \theta_2 < 1$$

$$\psi^{(3)}(x)=\frac{2}{x^3}+\frac{3}{x^4}+\frac{2}{x^5}-\frac{\theta_3}{x^7}, \qquad 0 < \theta_3 < 1$$

These asymptotic formulas were used to prove the first case when n=1, while we will prove the results mentioned above for all cases where $y = 1,2,\ldots$.

The Results

Theorem1 for $x > 0$ and $n = 1, 2, \ldots$ the following function is increasing.

$$\frac{\psi^{(\mathrm{n})}(x)}{\psi^{(\mathrm{n+1})}(x)} + \frac{x}{n} \qquad (4.1)$$

Proof. This means, showing that:

$$(n+1)(\psi^{(\mathrm{n+1})}(x))^2 - n\,\psi^{(\mathrm{n})}(x)\,\psi^{(\mathrm{n+2})}(x) > 0 \qquad (4.2)$$

Using the relation (3.9) and (3.10), we can write the next form:

$$\psi^{(n)}(x) = \sum_{i=0}^{2r-1} \frac{(-1)^{n+i}\,(n+i-1)B_i}{(i)!\,x^{n+i}} + \theta\frac{(-1)^{n+1}(n+2r-1)!\,B_{2r}}{(2r)!\,x^{n+2r}}, \qquad 0 < \theta < 1$$

for $n = 2m + 1$ (i.e. odd number); $m = 0, 1, 2, \ldots and\ r = 2$ we get

$$\psi^{(n)}(x) = \psi^{(2m+1)}(x) = \sum_{i=0}^{3} \frac{(-1)^{2m+2+i}\,(2m+i)B_i}{(i)!\,x^{2m+1+i}} + \theta\frac{(2m+4)!}{30(4)!\,x^{2m+5}}, \qquad 0 < \theta < 1$$

$$\psi^{(2m+1)}(x) < \frac{(2m)!}{x^{2m+1}} + \frac{(2m+1)!}{2x^{2m+2}} + \frac{(2m+3)!}{12x^{2m+3}}$$

$$= \frac{(2m)!}{12}x^{-2m-3}[(2m + 1)(2m + 2) + 6(2m + 1)x + 12x^2] \qquad (4.3)$$

Also

$$\psi^{(\mathrm{n+1})}(x) = \psi^{(2m+2)}(x) = \sum_{i=0}^{3} \frac{(-1)^{2m+3+i}\,(2m+1+i)B_i}{(i)!\,x^{2m+2+i}} + \theta\frac{(2m+5)!}{30(4)!\,x^{2m+6}}, \qquad 0 < \theta < 1$$

$$\psi^{(2m+2)}(x) < \frac{-(2m)!}{12}x^{-2m-4}[(2m + 1)(2m + 2)(2m+3) + 6(2m + 1)(2m+2)x + (2m+1)12x^2] \qquad (4.4)$$

And

$$\psi^{(\mathrm{n+2})}(x) = \psi^{(2m+3)}(x) = \sum_{i=0}^{3} \frac{(-1)^{2m+4+i}\,(2m+2+i)B_i}{(i)!\,x^{2m+3+i}} + \theta\frac{(2m+6)!}{30(4)!\,x^{2m+7}}, \qquad 0 < \theta < 1$$

$$\psi^{(2m+3)}(x) < \frac{-(2m)!}{12}x^{-2m-5}[(2m + 1)(2m + 2)(2m+3)(2m+4) + 6(2m + 1)(2m+2)(2m+3)x + (2m+1)(2m+2)12x^2] \qquad (4.5)$$

From (4.3), (4.4) and (4.5) we have:

$$w(x) = (2m+1)\psi^{(2m+1)}(x)\,\psi^{(2m+3)}(x) - (2m+2)\left(\psi^{(2m+2)}(x)\right)^2$$
$$< \frac{-1}{18}(m+1)(2m+1)^2(2m!)^2x^{-4(m+2)}[2m^2 + m(6x + 5) + 3x(x + 3) + 3]$$

Then $w(x) < 0$ for all $m = 0,1,2, \ldots$ yields (4.2) when n add number.

Now for $n = 2m$ (i.e. even number); $m = 0, 1, 2, \ldots and\ r = 1$ and by the same way as in the proof, we get

$$\psi^{(n)}(x) = \psi^{(2m)}(x) = \sum_{i=0}^{1} \frac{(-1)^{2m+1+i}\,(2m+i-1)B_i}{(i)!\,x^{2m+i}} + \theta\frac{(2m+1)!}{6(2)!\,x^{2m+2}}, \qquad 0 < \theta < 1$$

$$\psi^{(2m+)}(x) < \frac{-(2m-1)!}{x^{2m}} - \frac{(2m)!}{2x^{2m+1}}$$

$$= \frac{1}{2}(2m-1)!\,x^{-2m-1}(-2m-2x) \qquad (4.6)$$

Also

$$\psi^{(\mathrm{n+1})}(x) = \psi^{(2m+1)}(x) = \sum_{i=0}^{1} \frac{(-1)^{2m+2+i}\,(2m+i)B_i}{(i)!\,x^{2m+2+i}} + \theta\frac{(2m+2)!}{6(2)!\,x^{2m+3}}, \qquad 0 < \theta < 1$$

$$\psi^{(2m+1)}(x) > \frac{(2m)!}{x^{2m+1}} - \frac{(2m+1)!}{2x^{2m+2}}$$

$$= \frac{1}{2}2m\,(2m-1)!\,x^{-2m-2}((2m+1)+2 \qquad (4.7)$$

And

$$\psi^{(\mathrm{n+2})}(x) = \psi^{(2m+2)}(x) = \sum_{i=0}^{1} \frac{(-1)^{2m+3+i}\,(2m+1+i)B_i}{(i)!\,x^{2m+2+i}} + \theta\frac{(2m+3)!}{6(2)!\,x^{2m+4}}, \qquad 0 < \theta < 1$$

$$\psi^{(2m+3)}(x) < \frac{-(2m)!}{12}x^{-2m-5}[(2m + 1)(2m + 2)(2m+3)(2m+4) + 6(2m + 1)(2m+2)(2m+3)x + (2m+1)(2m+2)12x^2] \qquad (4.8)$$

From (4.6), (4.7) and (4.8) we have:

$$w(x) = (2m)\psi^{(2m)}(x)\,\psi^{(2m+2)}(x) - (2m+1)\left(\psi^{(2m+1)}(x)\right)^2$$

$$\begin{aligned} w(x) &< m^2(2m + 1)\big((2m - 1)!\big)^2(-2m - 2x - 2)(-2m - 2x)x^{-4m-4} \\ &\quad -m^2(2m + 1)\big((2m - 1)!\big)^2 x^{-4m-4}(2m + 2x + 1)^2 \\ &= -m2(2m + 1)x - 4(m+1)\Gamma(2m)^2 \end{aligned}$$

Then $w(x) < 0$ for all $m = 0,1,2,\ldots$ yields (4.2) when n even number. ☐

We will now use the monotonic properties that have been proven for the function $\frac{\psi^{(\mathrm{n})}(x)}{\psi^{(\mathrm{n}+1)}(x)} + \frac{x}{2n}$. These results enable us to obtain their exact limits through the next proof.

Theorem 2 for $x > 0$ *and* $n = 1,2,\ldots$ The following inequality is hold

$$0 < \frac{\psi^{(\mathrm{n})}(x)}{\psi^{(\mathrm{n}+1)}(x)} + \frac{x}{n} < \frac{1}{2n} \qquad (4.9)$$

Proof. When $n = 1$

$$\frac{\psi^{(1)}(x)}{\psi^{(2)}(x)} + x = \frac{\psi^{(1)}(x+1) + \frac{1}{x^2}}{\psi^{(2)}(x+1) - \frac{2}{x^3}} + x$$

We have

$$\lim_{x\to 0}\frac{x^3\,\psi^{(1)}(x+1) + x}{x^3\,\psi^{(2)}(x+1) - 2} + x = 0$$

also

$$\frac{\psi^{(1)}(x)}{\psi^{(2)}(x)} + x = \frac{\frac{1}{x} + \frac{1}{2x^2} + O(x^{-3})}{\frac{-1}{x^2} - \frac{1}{x^3} + O(x^{-4})} + x$$

and

$$\lim_{x\to\infty}\frac{\frac{1}{x} + \frac{1}{2x^2} + O(x^{-3})}{\frac{-1}{x^2} - \frac{1}{x^3} + O(x^{-4})} + x = \lim_{x\to\infty}\frac{\frac{-1}{2x^2} + O(x^{-3})}{\frac{-1}{x^2} - \frac{1}{x^3} + O(x^{-4})} = \frac{1}{2}$$

When $n = 2$

$$\frac{\psi^{(2)}(x)}{\psi^{(3)}(x)} + \frac{x}{2} = \frac{\psi^{(2)}(x+1) - \frac{2}{x^3}}{\psi^{(3)}(x+1) + \frac{6}{x^4}} + \frac{x}{2}$$

we have

$$\lim_{x\to 0}\frac{x^4\psi^{(2)}(x+1) - 2x}{x^4\,\psi^{(3)}(x+1) + 6} + \frac{x}{2} = 0$$

also

$$\frac{\psi^{2)}(x)}{\psi^{(3)}(x)} + \frac{x}{2} = \frac{\frac{-1}{x^2} + \frac{-1}{x^3} + O(x^{-4})}{\frac{2}{x^3} + \frac{3}{x^4} + O(x^{-5})} + \frac{x}{2}$$

and

$$\lim_{x\to\infty}\frac{\frac{-1}{x^2} + \frac{-1}{x^3} + O(x^{-4})}{\frac{2}{x^3} + \frac{3}{x^4} + O(x^{-5})} + \frac{x}{2} + x = \lim_{x\to\infty}\frac{\frac{1}{2x^3} + O(x^{-4})}{\frac{2}{x^3} + \frac{3}{x^4} + O(x^{-5})} = \frac{1}{4}$$

In general form:

$$\frac{\psi^{(\mathrm{n})}(x)}{\psi^{(\mathrm{n}+1)}(x)}+\frac{x}{n}=\frac{\psi^{(\mathrm{n})}(x+1)-\frac{(-1)^n n!}{x^{n+1}}}{\psi^{(\mathrm{n}+1)}(x+1)-\frac{(-1)^{n+1}(n+1)!}{x^{n+2}}}+\frac{x}{n}$$

We have

$$\lim_{x\to 0}\frac{x^{n+2}\,\psi^{(\mathrm{n})}(x+1)-(-1)^n n!\,x^n}{x^{n+2}\,\psi^{(\mathrm{n}+1)}(x+1)-(-1)^{n+1}(n+1)!}+\frac{x}{n}=0$$

also

$$\lim_{x\to\infty}\frac{\psi^{(\mathrm{n})}(x)}{\psi^{(\mathrm{n}+1)}(x)}+\frac{x}{n}=\lim_{x\to\infty}\frac{\frac{(-1)^n(n-1)!}{x^n}-\frac{(-1)^n n!}{2x^{n+1}}+O\left(x^{-(n+2)}\right)}{\frac{(-1)^n n!}{x^{n+1}}-\frac{(-1)^{n+1}(n+1)!}{2x^{n+2}}+O\left(x^{-(n+3)}\right)}+\frac{x}{n}$$

and

$$\lim_{x\to\infty}\frac{\frac{(-1)^n(n-1)!}{x^n}-\frac{(-1)^n n!}{2x^{n+1}}+O\left(x^{-(n+2)}\right)}{\frac{(-1)^n n!}{x^{n+1}}-\frac{(-1)^{n+1}(n+1)!}{2x^{n+2}}+O\left(x^{-(n+3)}\right)}+\frac{x}{n}=\lim_{n\to\infty}\frac{-\frac{(-1)^n(n)!}{2x^{n+1}}-\frac{(-1)^{n+1}(n+1)!}{2nx^{n+1}}+O\left(x^{-(n+2)}\right)}{\frac{(-1)^n n!}{x^{n+1}}-\frac{(-1)^{n+1}(n+1)!}{2x^{n+2}}+O\left(x^{-(n+3)}\right)}$$

$$=\frac{-\frac{(-1)^n(n)!}{2}-\frac{(-1)^{n+1}(n+1)!}{2n}}{(-1)^n(n)!}$$

$$=\begin{cases}\dfrac{\frac{(n)!}{2}-\frac{(n+1)!}{2n}}{-(n)!}=\dfrac{1}{2n} & if\ n\ odd\\[2ex] \dfrac{\frac{-(n)!}{2}+\frac{(n+1)!}{2n}}{(n)!}=\dfrac{1}{2n} & if\ n\ evev\end{cases}\qquad\square$$

Then the function $\frac{\psi^{(\mathrm{n})}(x)}{\psi^{(\mathrm{n}+1)}(x)}+\frac{x}{n}$ increases from 0 to $\frac{x}{2n}$ as x increases from 0 to ∞.

Example

$$\frac{\psi^{(1)}(x)}{\psi^{(2)}(x)}+x,\qquad\frac{\psi^{(2)}(x)}{\psi^{(3)}(x)}+\frac{x}{2},\qquad\frac{\psi^{(3)}(x)}{\psi^{(4)}(x)}+\frac{x}{3},\qquad\frac{\psi^{(4)}(x)}{\psi^{(5)}(x)}+\frac{x}{4},\qquad\frac{\psi^{(5)}(x)}{\psi^{(6)}(x)}+\frac{x}{5}$$

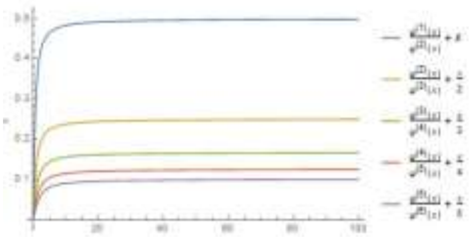

Figure 1: Special functions of polygamm functions.

As sequence of the above result, we obtain next property.

Corollary 3 for $x > 0\ and\ n = 1,2,\ldots$ the function $\frac{\psi^{(\mathrm{n})}(x)}{\psi^{(\mathrm{n}+1)}(x)}+\frac{x}{n}$ has horizontal asymptotic line $y=\frac{1}{2n}$.

Conclusion

This paper presents important properties of polygamma functions, namely explaining the increasing behaviour of these functions, and presenting their upper and lower bounds.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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