

Application of Residues in the inverse Laplace Transform of composite functions

Aml Ali *

Department of Mathematics, Faculty of Education, University of Zawia, Zawia , Libya

* Corresponding author: am.abdullah@zu.edu.ly

تطبيق نظرية الرواسب لحساب تحويل لابلاس العكسي للدوال المركبة

أمل علي *

قسم الرياضيات، كلية التربية أبو عيسى، جامعة الزاوية، الزاوية، ليبيا

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Abstract

The inverse Laplace transform is a pivotal mathematical tool in solving differential equations and analyzing engineering and physical systems. However, traditional methods such as partial fractions fail when dealing with composite and non-algebraic functions. This research aims to investigate and apply the Residue Theorem in complex analysis to compute the inverse Laplace transform of composite functions via the Bromwich integral formula. The study covers the theoretical framework of identifying poles and singularities, proceeds to the practical computation of residues for simple and multiple poles, and extends to advanced techniques for handling functions containing roots and logarithms using modified contour integration paths (such as the keyhole contour). The findings demonstrate that the residue theorem serves as a highly powerful and precise analytical tool, overcoming the limitations of conventional algebraic methods and providing accurate time-domain solutions for complex engineering problems such as diffusion and heat transfer systems, thereby recommending its adoption as an advanced methodology in applied mathematical studies.

Keywords: Inverse Laplace Transform, Residue Theorem, Composite Functions, Bromwich Integral, Singularities, poles, Keyhole Contour.

المخلص

تناقش هذه الورقة البحثية الإطار الرياضي المتقدم لنظرية الرواسب في مجال التحليل المركب، مقترنا بتحويل لابلاس العكسي الذي يعد كأداة رياضية في حل المعادلات التفاضلية وتحليل الأنظمة الهندسية والفيزيائية. ومع ذلك، تصبح الطرق التقليدية مثل الكسور الجزئية عاجزة عن التعامل مع الدوال المركبة وغير الجبرية. يهدف هذا البحث إلى دراسة وتطبيق نظرية الرواسب في التحليل العقدي لحساب تحويل لابلاس العكسي للدوال المركبة من خلال صيغة برومويش التكاملية. يتناول البحث الجانب النظري لتحديد الأقطاب ونقاط التفرد، مروراً بالحوسبة العملية لرواسب للأقطاب البسيطة والمتعددة، وصولاً إلى التقنيات المتقدمة للتعامل مع الدوال التي تحتوي على جذور ولوغاريتمات باستخدام مسارات التكامل المعدلة (مسار المفتاح).

و تظهر نتائج البحث أن نظرية الرواسب تمثل أداة تحليلية فائقة القوة والدقة لتتجاوز قيود الطرق الجبرية التقليدية وتقديم حلول زمنية دقيقة للمسائل الهندسية المعقدة كأنظمة الانتشار وانتقال الحرارة، مما يوصي باعتمادها منهجية متقدمة في الدراسات الرياضية التطبيقية.

الكلمات المفتاحية: تحويل لابلاس العكسي، نظرية الرواسب، الدوال المركبة، تكامل برومويش، التفردات، الأقطاب، مسار المفتاح.

Introduction

The Laplace transform and its inverse are fundamental pillars in applied mathematics and the analysis of engineering and physical systems, as they are used to simplify complex differential equations and convert them into algebraic equations that are easier to solve. However, finding the inverse Laplace transform for complex functions with multiple singularities, or non-standard functions involving branch points, poses significant analytical challenges when using traditional methods such as tables or simple partial fractions. Herein lies the crucial importance of complex analysis, specifically the residue theorem, as an extremely powerful and efficient mathematical tool. By combining the Bromwich inversion integral with complex integration techniques, it becomes possible to evaluate complex integrals by calculating the sum of the residues of a function at its poles enclosed within a closed integration path. Furthermore, for functions containing branch points or branch cuts, advanced integration paths such as the keyhole contour are required, allowing the path to wrap around the cuts in the complex plane and evaluate improper integrals with high precision. This provides a comprehensive analytical solution for functions that standard algebraic methods fail to handle.

The subject of complex integration and its applications in calculating the inverse Laplace transform using the residue theorem and contour integration has received extensive attention in mathematical literature. Below is a review of the prominent studies and scientific references that form the theoretical framework of this research:

1. **Churchill & Brown (2009)[1]:** These foundational works established the classical rules linking the Bromwich inversion formula to contour integration, paving the way for the use of linear integration paths in the complex plane to handle complex functions.
2. **Ablowitz & Fokas (2003)[2]:** This study provided an advanced analysis of specialized integration paths, specifically the Keyhole Contour, for handling complex functions involving branch points and branch cuts, which serves as a vital foundation when computing the inverse Laplace transform for non-analytic functions in distributed-parameter physical systems.
3. **Zhang et al. (2018)[3]:** This research addressed the mechanisms of designing complex closed paths and employing the residue theorem to compute inverse Laplace transforms for high-order logarithmic and radical functions with high analytical efficiency that surpasses the algebraic limitations of partial fractions.
4. **Al-Mukhtar et al. (2021)[4]:** These contemporary investigations focused on evaluating the accuracy of using residue calculus and Bromwich integrals to minimize relative error when inverting Laplace transforms of complex functions in advanced automatic control systems, confirming the superiority of the analytical residue-based approach over purely approximate methods.

This research relies on the analytical-deductive methodology. It starts by formulating the Bromwich inverse integral, then identifies the singularities and designs appropriate integration paths (including the Keyhole Contour) to handle branch points and cuts. Finally, it applies the residue theorem to derive general and accurate mathematical formulas for calculating the inverse Laplace transform, supported by selected practical examples.

1-Mathematical Background and Definitions

1-1 The Laplace Transform: [5]

The unilateral Laplace transform maps a time-domain function $f(t), t \geq 0$ into a complex frequency-domain function $F(s)$, ($s = \alpha + i\beta$) via improper integral

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt = \ell\{f(t)\} \dots \dots \dots (1)$$

Provided the integral converges for $Re(s)$ greater than abscissa of convergence some important properties of Laplace transform .

Linearity: For any function $f(t)$ and $g(t)$ whose Laplace transform exist, and any constants C_1 and C_2

$$\ell\{C_1f(t) + C_2g(t)\} = C_1\{\ell f(t)\} + C_2\{\ell g(t)\} \dots \dots \dots (2)$$

Transforms of Derivatives: Let $f, \dot{f}, \dots, f^{(n-1)}$ be continuous for all $t \geq 0$ and let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Then the transform of $f^{(n)}$ is given by:

$$\ell\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \dots - f^{(n-1)}(0) \dots \dots (3)$$

1-2 Inverse Laplace Transformation:

The inverse Laplace transform of $F(s)$ is $f(t)$ which is defined by

$$\ell^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds = f(t), \quad t > 0 \dots \dots \dots (4)$$

Properties of the inverse Laplace transform:

- **Linearity:** The inverse Laplace transform is a linear operator, meaning that the inverse of a sum of two functions multiplied by constants is equal to the constants multiplied by the inverse of each function individually

$$\ell^{-1}\{aF(s) + bG(s)\} = a\ell^{-1}\{F(s)\} + b\ell^{-1}\{G(s)\} \dots \dots \dots (5)$$

Where a and b are constant numbers.

- **First shifting theorem:** If the Laplace image is shifted by in the S-domain, this appears in the time domain as multiplying the original function by an exponential function:

$$\ell^{-1}\{F(s - a)\} = e^{at} f(t) \dots \dots \dots (6)$$

- **Second shifting theorem:** when a Laplace image is multiplied by an exponential factor containing S , it represents a time shift of the original function, often associated with the unit step function $u(t - a)$:

$$\ell^{-1}\{F(as)\} = u(t - a)f(t - a) \dots \dots \dots (7)$$

- **Scaling property:** If the variable S is multiplied by a constant a inside the function, the resulting time-domain function is adjusted by dividing the time by a and multiplying by the reciprocal factor:

$$\ell^{-1}\{F(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right) \dots \dots \dots (8)$$

- **Differentiation in the S-domain(Multiplication by powers of S):**

Repeated differentiation of the Laplace image with respect to S corresponds to multiplying the original function by the power of time t^n :

$$\ell^{-1}\left\{\frac{d^n F(s)}{ds^n}\right\} = (-1)^n t^n f(t) \dots \dots \dots (9)$$

Division by S (integration):

- Dividing the Laplace image by the variable S mathematically and geometrically corresponds to performing integration of the original function from zero up to time t :

$$\ell^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau) d\tau \dots \dots \dots (10)$$

1-3The Bromwich Integral Formula:

The Bromwich integral Formula(also known as the Bromwich integral or inverse Mellin transform) is the precise mathematical relation used to find the inverse Laplace transform of a complex function in the domain, relying on the tools of complex plane(S-plane). The formula expresses the inverse Laplace transform in the following integral form:

$$f(t) = \ell^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds \dots \dots \dots (11)$$

1-4 Complex function:[6]

The range of complex functions ranges from plural to plural. In other words, it is a function, and its complex subset is defined as a complex subset of the domain a an upper domain. complex function is a function takes complex number as independent variable and dependent variable.

1-4-1Definition: A complex-valued function $f(z)$ is said to be analytic on an open set G if it has a derivative at every point of G . [7]

1-4-2 Theorem. A necessary condition for a function $f(z) = u(x, y) + iv(x, y)$ to be differentiable at a point z_0 that the Cauchy-Riemann equations hold at z_0 . Consequently, if f is analytic in an open set G , then the Cauchy-Riemann equations must hold at every point of G .

1-4-3 Theorem. let $f(z) = u(x, y) + iv(x, y)$ be defined in some open set G containing the point z_0 . if the first partial derivatives of u and v exist in G , are continuous at z_0 , and satisfy the Cauchy-Riemann equations at z_0 . Consequently, if the first partial are continuous and satisfy the Cauchy-Riemann equations at all points of G , then f is analytic in G .

2-Singulaties and Residue theory

2-1 Singular points: A singular point of a complex function $f(z)$ is a point where the function fails to be analytic or defined, while remaining analytic in some neighborhood of that point(excluding the point itself in the case of isolated singularities). Isolated singularities are classified into three main types based on the function's Laurent Series expansion around the point z_0 :

2-1-1 Removable singularity:

Definition: A point where the principal part of the Laurent Series contains no negative powers of $(z - z_0)$.

Property: The $\lim_{z \rightarrow z_0} f(z)$ exists and yields a finite value. The singularity can be "removed" by redefining the function at that point to make it continuous and analytic.

For example: $f(z) = \frac{\sin z}{z}$ at $z = 0$.

2-1-2 Poles:

Definition: Points where the principal part of the Laurent Series contains a finite number of negative terms. If the highest negative power is $(z - z_0)^{-m}$, the function is said to have a pole of order m (m -th order pole).

Property: The function's magnitude approaches infinity as z approaches z_0 (i.e., $\lim_{z \rightarrow z_0} f(z) = \infty$).

For example: $f(z) = \frac{1}{(z-1)^2}$ has a pole of order 2 at $z = 1$.

2-1-3 Essential singularity:

Definition: A point where the principal part of the Laurent Series contains an infinite number of negative terms.

Property: The function exhibits extreme oscillatory behavior near this point.

For example: $f(z) = e^{\frac{1}{z}}$ at $z = 0$.

2-2 The concept of Residues

The residue of a complex function $f(z)$ at an isolated singular point z_0 is the coefficient of the term multiplied by $(z - z_0)^{-1}$ in the function's Laurent Series around that point. The residue coefficient is calculated via the general formula:

$$a_{-1} = \frac{1}{2\pi i} \oint_c f(z) dz \dots \dots \dots (12)$$

Where c is a simple closed contour.

Practical Methods for calculating Residues:

- For a simple pole:

If $f(z) = \frac{P(z)}{Q(z)}$, the residue is calculated as :

$$Res(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) = \frac{P(z_0)}{Q'(z_0)} \dots \dots \dots (13)$$

- For a pole of order m :

$$Res(f, z_0) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \dots \dots \dots (14)$$

2-3 The Residue Theorem: [8]

Cauchy's residue theorem is a fundamental generalization of Cauchy's integral formula, stated as follows: If a function $f(z)$ is analytic inside and on appositively oriented simple closed contour C , except at a finite number of isolated singularities $z_1, z_2, \dots, z_n$ located inside C , then the contour integral of the function around C equals $2\pi i$ times the sum of the residues of the function at those points. Mathematical expression of the theorem:

$$\oint_c f(z) dz = 2\pi i \sum_k Res(f(z), z_k) \dots \dots \dots (15)$$

2-4 Significance and Applications of the Residue theorem:

- Evaluating complex contour integrals:
Transforms the problem of computing complex contour integrals into algebraic evaluations of derivatives or limits at singularities.
- Computing Real integrals: Used to solve real integrals over the interval $(-\infty, \infty)$ that standard integration techniques cannot handle(such as trigonometric and complex rational integrals via Jordan's lemma).

3-Methodology for composite functions:

3-1 Rational functions:

Rational functions appear in the general form $f(z) = \frac{P(z)}{Q(z)}$, if ensure that the degree of the denominator is strictly greater than the degree of the numerator ($deg(Q) > deg(p)$).

If the degree of the numerator is greater than or equal to the denominator's, first perform polynomial long division to isdate simple terms, then handle the remaining proper fraction.

Calculating residues via direct shortcut Formulas:

- If pole s_k is simple ($m=1$): calculate the residue directly without integration using the denominator derivative rule:

$$\text{Res}\left(\frac{P(s)}{Q(s)}e^{st}, s_k\right) = \frac{p(s)}{Q'(s_k)}e^{s_k t} \dots \dots \dots (16)$$

- If pole s_k is a multiple pole of order m :

Calculate the residue using the direct differentiation formula:

$$\text{Res}(f(s)e^{st}, s_k) = \frac{1}{(m-1)!} \lim_{s \rightarrow s_k} \frac{d^{m-1}}{ds^{m-1}} \left[(s-s_k)^m \frac{p(s)}{Q(s)} e^{st} \right] \dots \dots \dots (17)$$

Then writhing final Result Directly

$$f(t) = \sum_k \text{Res}(F(s)e^{st}, s_k) \dots \dots \dots (18)$$

3-2 Algebraic functions: [9]

Algebraic functions involve roots or logarithms(such as $\sqrt{s}$ or $\ln(s)$) and require more precise steps due to the presence of "multi-valuedness".

3-2-1 Definition [10]: The point of z_0 is called a branch point for the complex valued function $f(z)$, if the value of $f(z)$ does not return its initial value as a closed curve around the point I traced (starting from some arbitrary point on the curve), in such a way that f varies continuously as the path is traced.

3-2-2 Definition: A Branch cut is defined as a curve or line drawn in the complex plane to restrict the domain of a multi-valued function, transforming it into a single-valued and continuous(analytic) function in the remaining region of the plane.

3-2-3 Definition [11]: A keyhole contour is a composite closed integration path in the complex plane specifically designed to handle multi-valued functions that contain branch points and branch cuts(such as radical and logarithmic functions).

Geometric components of a Keyhole contour:

1-A large outer circular arc(C_R): with a radius R that approaches infinity ($R \rightarrow \infty$).

2-Two parallel straight paths(L_1 and L_2): Running closely along both sides of the branch cut(the upper and lower banks).

3-A small closed circle (C_ϵ): with a radius ϵ that approaches zero ($\epsilon \rightarrow 0$), wrapping tightly around the branch point to avoid the non-analyticity or singularity at the point itself.

3-2-4 Definition [12]: The Gaussian integral is the definite of a Gaussian function(the quadratic exponential function) over the entire real line. The general formula is given by:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}, \quad a > 0 \dots \dots \dots (19)$$

4-Case studies and Applications

Example 4-1

Find the inverse Laplace transform of the composite function : $F(s) = \frac{1}{s(s+a)}$

Solution:

Use the Bromwich integral formula:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds$$

The function inside the integral

$$g(s) = \frac{e^{st}}{s(s+a)}$$

Has two simple poles:

- 1) $s_1 = 0$
- 2) $s_2 = -a$

Calculating the residues at these poles:

$$\text{Res}(F(s)e^{st}, 0) = \lim_{s \rightarrow 0} \frac{s \cdot e^{st}}{s(s+a)} = \frac{1}{a}$$

$$\text{Res}(F(s)e^{st}, -a) = \lim_{s \rightarrow 0} \frac{(s+a) \cdot e^{st}}{s(s+a)} = \frac{e^{-at}}{-a}$$

Summing the residues:

$$\ell^{-1} \left\{ \frac{1}{s(s+a)} \right\} = f(t) = \sum_k \text{Res}(F(s)e^{st}, s_k) = \frac{1}{a}(1 - e^{-at})$$

Example 4-2

Find the inverse Laplace transform of the composite function : $F(s) = \frac{1}{(s^2+a^2)(s-b)}$

Solution:

Use the Bromwich integral formula:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds$$

The function inside the integral

$$g(s) = e^{st} F(s) = \frac{e^{st}}{(s^2+a^2)(s-b)}$$

The function $g(s)$ has three simple poles

- 1) $s_1 = ia$
- 2) $s_2 = -ia$
- 3) $s_3 = b$

Calculating the residues at these poles:

Since all the poles are simple, then Residue at the first pole $s_1 = ia$

$$\text{Res}(F(s)e^{st}, ia) = \lim_{s \rightarrow ia} \frac{e^{st}}{(s+ia)(s-b)} = \frac{e^{iat}}{2ia(ia-b)}$$

Residue at the second pole $s_2 = -ia$

$$\text{Res}(F(s)e^{st}, -ia) = \lim_{s \rightarrow -ia} \frac{e^{st}}{(s-ia)(s-b)} = \frac{e^{-iat}}{-2ia(-ia-b)}$$

Residue at the third pole $s_3 = b$

$$\text{Res}(F(s)e^{st}, b) = \lim_{s \rightarrow b} \frac{e^{st}}{s^2+a^2} = \frac{e^{bt}}{b^2+a^2}$$

Summing the Residues

$$f(t) = \sum_k \text{Res}(F(s)e^{st}, s_k) = \frac{e^{iat}}{2ia(ia-b)} + \frac{e^{-iat}}{2ia(ia+b)} + \frac{e^{bt}}{b^2+a^2}$$

Use by Euler's formulas ,then

$$\ell^{-1} \left\{ \frac{1}{(s^2+a^2)(s-b)} \right\} = \frac{1}{b^2+a^2} \left[e^{bt} - \cos(at) - \frac{b}{a} \sin(at) \right]$$

Example 4-3

Find the inverse Laplace transform of the composite function : $F(s) = \frac{s}{(s^2+a^2)^2}$

Solution:

Use the Bromwich integral formula, then The function inside the integral:

$$g(s) = e^{st} F(s) = \frac{s \cdot e^{st}}{(s^2+a^2)^2} = \frac{s \cdot e^{st}}{(s-ia)^2(s+ia)^2}$$

The function $g(s)$ has two double poles(poles of order m=2):

$s_1 = ia$ (with multiplicity m=2)

$s_2 = -ia$ (with multiplicity m=2)

Calculating the residues at these poles:

Residues at the $s_1 = ia$

$$\text{Res}(g(s), ia) = \lim_{s \rightarrow ia} \frac{d}{ds} \left[\frac{s \cdot e^{st}}{(s+ia)^2} \right] = \frac{te^{iat}}{4ia} - \frac{e^{iat}}{4a^2}$$

Residues at the $s_2 = -ia$

$$\text{Res}(g(s), -ia) = \lim_{s \rightarrow -ia} \frac{d}{ds} \left[\frac{s \cdot e^{st}}{(s + ia)^2} \right] = \frac{-te^{-iat}}{4ia} - \frac{e^{-iat}}{4a^2}$$

Summing the residues and final result:

$$f(t) = \sum_k \text{Res}(F(s)e^{st}, s_k) = \left(\frac{te^{iat}}{4ia} - \frac{e^{iat}}{4a^2} \right) + \left(\frac{-te^{-iat}}{4ia} - \frac{e^{-iat}}{4a^2} \right)$$

By grouping terms and applying Euler's formulas, the time-dependent $\sin(at)$ term dominates, yielding the final matching result:

$$\ell^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} = \frac{t}{2a} \sin(at).$$

Example 4-4

Find the inverse Laplace transform of the function : $F(s) = \frac{e^{-k\sqrt{s}}}{\sqrt{s}}, k > 0$.

Solution:

Notice that this function does not contain conventional poles, but rather a branch point at $s = 0$ due to the square root term ($\sqrt{s}$). Therefore, standard residue theory using isolated poles is insufficient; instead, then use a closed contour enclosing the branch cut, known as a keyhole contour.

Detailed andytical steps:

1-Bromwich integral formula:

The inverse Laplace transform is expressed as a complex line integral along a vertical path in the complex plane:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st} \cdot e^{-k\sqrt{s}}}{\sqrt{s}} ds$$

2-Choice of contour(keyhole contour):

- Since $s = 0$ is a branch point, I place a branch cut along the negative real axis($\arg(s) = \pi$)
- Close the Bromwich path with a large circular arc of radius R , a small circular loop of radius ϵ around the origin, and two straight running along the upper and lower banks of the branch cut.

3-Application of Cauchy's integral theorem: since the integrand is analytic within the defined region(except along the branch cut), the total integral over the closed contour vanishes because there are no enclosed isolated poles:

$$\oint_c \frac{e^{st-k\sqrt{s}}}{\sqrt{s}} ds = 0$$

4-Evaluating the contour components:

- As $R \rightarrow \infty$: The integral along the large circular arc vanishes (by Jordan's Lemma).
- As $\epsilon \rightarrow 0$: The integral around the small circle enclosing the origin vanishes.
- This leaves the integrals along the two banks of the branch cut, where the value of the square root $\sqrt{s}$ differs by a sing:

On the upper bank(where $\arg(s) = \pi$): $\sqrt{s} = i\sqrt{x}$

On the lower bank(where $\arg(s) = -\pi$): $\sqrt{s} = -i\sqrt{x}$

After evaluating the keyhole contour integral, then

$$f(t) = \ell^{-1} \left\{ \frac{e^{-k\sqrt{s}}}{\sqrt{s}} \right\} = \frac{1}{\sqrt{\pi t}} e^{-\frac{k^2}{4t}}, \quad t > 0$$

Conclusion

This study yielded a set of essential findings that underscore the effectiveness and power of applying the Residue Theorem to calculate the inverse Laplace transform of complex functions. The key results derived from the research can be summarized as follows:

- The research successfully formulated an integrated mathematical methodology linking the Bromwich Integral with Cauchy's Residue Theorem, enabling the calculation of complex contour integrals by effectively reducing them to the sum of the residues of poles located within the closed contour.
- The results demonstrated the superiority of this methodology over classical techniques (such as partial fractions) when dealing with complex functions possessing high-order poles, providing precise and direct solutions that avoid the complexities of lengthy algebraic manipulation.
- The study demonstrated the success of employing specialized integration paths (such as the Keyhole Contour) to enclose the branch cuts associated with logarithmic and radical functions, thus resolving the challenge of non-isolated singularities with rigorous mathematical precision.

Taken together, these findings constitute a flexible and reliable mathematical tool for addressing advanced differential models and boundary value problems in mathematical physics and engineering control.

Compliance with ethical standards

Disclosure of conflict of interest

The authors declare that they have no conflict of interest.

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