Solution of the Cubic Duffing Equation via the Differential Transform Method

Najla A. Amazib [1], Fadwa A. M. Madi [2], Fawzi A. Abdelwahid [3*]
[1,2,3] Department of Mathematics, Faculty of Science, University of Benghazi, Benghazi, Libya
Corresponding author: fawzi.abdelwahid@uob.edu.ly

حل معادلة (Duffing) التكعيبية بواسطة طريقة التحويل التفاضلي

نجلاء السنوسي امعزب [1]، فدوى الصادق محمد ماضي [2] , فوزي إبريك عبد الواحد [3*]
[1،2،3] قسم الرياضيات، كلية العلوم، جامعة بنغازي، ليبيا

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Abstract:
In this work, we review the dimensional differential transform and introduce a formula for the cubic non-linear term $f(u) = u^3(t)$. Then, we use the differential transform method, with this formula, to solve the well-known cubic Duffing's equation. Next, with the help of Maple program tools, we show that the series solutions obtained by the differential transform method converge rapidly to the Alvaro and Jairo exact solutions, which given in term of the Jacobian elliptic functions. This shows that the differential transform method is capable to solve non-linear ordinary differential equations, when the non-linear term is given as a polynomial of degree higher than three.

Keywords: Ordinary differential equations, Deferential transform method, Jacobian Elliptic functions.

الملخص:
في هذا العمل، قمنا بمراجعة التحويل التفاضلي في بعد واحد (أحادي البعد) وقدمنا صيغة للحد الغير الخطي التكعيبي $f(u) = u^3(t)$ ثم استخدمنا طريقة التحويل التفاضلي، مع هذه الصيغة، لحل معادلة (دوفينغ) التكعيبية المعروفة. بعد ذلك، وبمساعدة أدوات برنامج (**Maple**)، نُبيّن أن الحلول المتسلسلة الناتجة عن طريقة التحويل التفاضلي تتقارب بسرعة نحو الحلول الدقيقة التي قدّمها **Alvaro** و **Jairo** ، والتي تُعطى بدلالة دوال (جاكوبي) الإهليلجية. وهذا يوضح أن طريقة التحويل التفاضلي قادرة على حل المعادلات التفاضلية العادية غير الخطية، عندما يُعطى الحد الغير الخطي على شكل كثير حدود من درجة أعلى من الثلاثة.

الكلمات المفتاحية: معادلات تفاضلية عادية ، طريقة التحويل التفاضلي، دوال جاكوبي الإهليلجية.

1. Introduction

Non-linear ordinary differential equations, whose solutions cannot be found explicitly, arise in essentially every branch of modern science, engineering and physics [1-3]. One of the important methods for solving non-linear ordinary differential equations is the differential transform method (DTM) [4-7]. The concept of differential transform (DT) first proposed by Zhou in 1986, [4] and then applied to linear and non-linear initial value problems [5-7]. Furthermore, the differential transform method is a semi analytical technique that used the Taylor series to construct the solutions of differential equations in the form of a power series. The cubic Duffing equation is an ordinary differential equation with third power nonlinearity and there are many problems in physics and engineering that lead to this type of nonlinear equation [1,8,9].

2. The Differential Transforms

Our aim in this section is to introduce a formula of the differential transform for the cubic non-linear term $f(u) = u^3(t)$. To find this formula, we review first the definition of the (DT) and some important properties and formulas [6].

Definition: (1)

Let $f(t)$ be a $C^{\infty}(I)$ function in an open interval I of R, and t_0 be any point of I, then the Taylor series expansion of $f(t)$ about t_0 is given by

$$f(t)=\sum_{k=0}^{\infty}\left[\frac{f^{(k)}(t)}{k!}\right]_{t=t_0}(t-t_0)^k \tag{1}$$

Note that: If the expansion (1) converges to $f(t)$ for every t in the neighborhood of t_0, then we say that $f(t)$ is analytic function at t_0.

Definition (2)

Let $f(t)$ be an analytic function at $t_0=0$, then the k^{th} differential transform of $f(t)$ is defined as

$$D_T\{f(t)\}:=\left[\frac{f^{(k)}(t_0)}{k!}\right]_{t_0=0} \tag{2}$$

In (2), $D_T\{f(t)\}$ usually denoted by $F(k)$ and it represents the one-dimensional (DT). And the inverse (DT) of $F(k)$ denotes by $D_T^{-1}\{F(k)\}$ and it define as

$$D_T^{-1}\{F(k)\}=f(t):=\sum_{k=0}^{\infty}F(k)\,t^k \tag{3}$$

Using the definition (2), we can prove the linearity property of the (DT) and establish the following important formulas [4-7].

Theorem: (1) (The DT of Derivatives)

Let $f(t)$ be an analytic function at $t_0=0$, with $D_T\{f(t)\}=F(k)$ then

$$D_T\{f^{(n)}(t)\}=\frac{(k+n)!}{k!}F(k+n) \tag{4}$$

Theorem: (2) (The DT of the Product of Two Functions)

Let $f_1(t)$ and $f_2(t)$ be analytic functions at t_0 with $D_T\{f_1(t)\}=F_1(k)$ and $D_T\{f_2(t)\}=F_2(k)$, then

$$D_T\left\{f_1(t)\cdot f_2(t)\right\} = \sum_{n=0}^{k} F_1(n)F_2(k-n), \tag{5}$$

Using theorem (2), we can write the following corollary:

Corollary (1): (The (DT) of a square non-linear term $f(u)=u^2(t)$)

Let $u(t)$ be an analytic function at $t_0 = 0$ with $D_T\{u(t)\}=U(k)$, then

$$D_T\left\{f(u)\right\} = D_T\left\{u^2(t)\right\} = \sum_{n=0}^{k} U(n)U(k-n) \tag{6}$$

Next, by using the corollary (1), we can introduce the following result

Theorem (3): (The (DT) of a cubic non-linear term $f(u)=u^3(t)$)

Let $u(t)$ be an analytic function at $t_0 = 0$ with $D_T\{u(t)\}=U(k)$, then

$$D_T\left\{f(u)\right\} = D_T\left\{u^3(t)\right\} = \sum_{n=0}^{k}\sum_{i=0}^{n} U(i)U(n-i)U(k-n) \tag{7}$$

To prove this result, we assume that $f_1(t)=u^2(t)$ and $f_2(t)=u(t)$. This implies that

$$D_T\left\{f_1(t)\right\} = D_T\left\{u^2(t)\right\} = \sum_{i=0}^{k} U(i)U(k-i) \tag{8}$$

$$D_T\{f_2(t)\} = D_T\{u(t)\} = U(k), \tag{9}$$

At the end, by using (8) and (9) with (5), we can establish the formula (7).

3. The Solutions Cubic Duffing Equation

In this section, we applied the differential transform method to the cubic Duffing equation

$$\ddot{u} + f(u) = 0, \qquad f(u) = \alpha u + \beta u^3, \qquad \beta \neq 0 \tag{10}$$

Subject to the initial conditions

$$u(0) = u_0, \qquad \dot{u}(0) = u_1. \tag{11}$$

In (11) $\dot{u}$ represents the time derivative and the nonlinearity term $f(u)$ is usually a cubic polynomial [8]. Then we compare our solutions with the exact solutions obtained by Alvaro and Jairo and Alvaro, which given in the form of the Jacobian elliptic functions, provided that $\alpha + u_0^2\beta \neq 0$ [8-10].

To find the solution of the initial value problem (8-9), we apply the (DT) on both sides of the equation (10). This implies

$$D_T\left\{\frac{d^2u(t)}{dt^2}\right\} = -\alpha D_T\{u(t)\} - \beta D_T\{u^3(t)\} \tag{12}$$

Then using the formula (4) and the result of theorem (3), we can write

$$(k+1)(k+2)U(k+2) = -\alpha U(k) - \beta\sum_{n=0}^{k}\sum_{i=0}^{n} U(i)U(n-i)U(k-n) \tag{13}$$

and the initial conditions (11) transform to

$$U(0) = u_0, \qquad U(1) = u_1 \tag{14}$$

This leads to the iteration formula

$$U(k+2) = \frac{-1}{(k+1)(k+2)}\left(\alpha U(k) + \beta\sum_{n=0}^{k}\sum_{i=0}^{n} U(i)U(n-i)U(k-n)\right), \quad k = 0,1,2,.. \tag{15}$$

At the end, using the iteration formula (15) with (14) for $k = 0,1,2,\ldots$, we can easily calculate $U(k)$ to any desire order, then the inverse (DT) will lead to series solution.

To compare the differential transform method with the Alvaro and Jairo approach, we follow [8], and solve the initial value problem (10-11) for different values of $\alpha\ \&\ \beta$.

Example (1): For $\alpha = \beta = 2$, $u_0 = 1$ and $u_1 = 0$, the transform of the initial conditions (14) and the iteration formula (15) reads

$$U(k+2)=\frac{-2}{(k+1)(k+2)}\left(U(k)+\sum_{n=0}^{k}\sum_{i=0}^{n}U(i)U(n-i)U(k-n)\right),\quad k=0,1,2,.. \tag{16}$$

$$U(0)=1\ ,\qquad U(1)=0 \tag{17}$$

Next, we use the iteration formula (16) with (17), we can calculate

$$\begin{aligned}U(2)&=-\left(U(0)+\sum_{n=0}^{0}\sum_{i=0}^{n}U(i)U(n-i)U(0-n)\right)\\&=-\left(U(0)+\sum_{i=0}^{0}U(i)U(0-i)U(0)\right)=-\left(U(0)+U^{3}(0)\right)=-2\end{aligned} \tag{18}$$

$$\begin{aligned}U(3)&=\frac{-2}{6}\left(U(1)+\sum_{n=0}^{1}\sum_{i=0}^{n}U(i)U(n-i)U(1-n)\right)\\&=\frac{-1}{3}\left(U(1)+\sum_{i=0}^{0}U(i)U(0-i)U(1)+\sum_{i=0}^{1}U(i)U(1-i)U(0)\right)\\&=\frac{-1}{3}\left(0+U(0)U(1)U(0)+U(1)U(0)U(0)\right)=0\end{aligned} \tag{19}$$

$$\begin{aligned}U(4)&=\frac{-2}{12}\left(U(2)+\sum_{n=0}^{2}\sum_{i=0}^{n}U(i)U(n-i)U(2-n)\right)\\&=\frac{-1}{6}\left(U(2)+\sum_{i=0}^{0}U(i)U(0-i)U(2)+\sum_{i=0}^{1}U(i)U(1-i)U(1)+\sum_{i=0}^{2}U(i)U(2-i)U(0)\right)\\&=\frac{-1}{6}\begin{pmatrix}U(2)+U(0)U(0)U(2)+0+U(0)U(2)U(0)+\\U(1)U(1)U(0)+U(2)U(0)U(0)\end{pmatrix}=\frac{-1}{6}\left(4U(2)\right)=\frac{4}{3}\end{aligned} \tag{20}$$

$$\begin{aligned}U(5)&=\frac{-2}{10}\left(U(3)+\sum_{n=0}^{3}\sum_{i=0}^{n}U(i)U(n-i)U(3-n)\right)\\&=\frac{-1}{5}\begin{pmatrix}U(3)+\sum_{i=0}^{0}U(i)U(-i)U(3)+\sum_{i=0}^{1}U(i)U(1-i)U(2)+\\\sum_{i=0}^{2}U(i)U(2-i)U(1)+\sum_{i=0}^{3}U(i)U(3-i)U(0)\end{pmatrix}\\&=\frac{-1}{5}\begin{pmatrix}0+0+U(0)U(1)U(2)+U(1)U(0)U(2)+U(0)U(3)U(0)+\\U(1)U(2)U(0)+U(2)U(1)U(0)+U(3)U(0)U(0)\end{pmatrix}=0\end{aligned} \tag{21}$$

Furthermore, the iteration formula (16) with (17) and making use of Maple-23 Package, enable us to calculate $U(k)$ for any reasonable desire order. Hence, for $k=4,5,6,7,8,9,10,11,12$, we found

$U(6)=\frac{-52}{45}$, $U(7)=0$, $U(8)=\frac{46}{45}$, $U(9)=0$, $U(10)=\frac{-1768}{2045}$,

$U(11)=0$, $U(12)=\frac{50156}{66825}$ $U(13)=0$ $U(14)=-\frac{561224}{868725}, \ldots\ldots$ (22)

This leads to the series solution

$$u(t)=1-2t^2+\frac{4}{3}t^4-\frac{52}{45}t^6+\frac{46}{45}t^8-\frac{1768}{2025}t^{10}+\frac{50156}{66825}t^{12}-\frac{561224}{868725}t^{14}+\ldots.., \quad (23)$$

This is identical to the series solution of the Alvaro and Jairo exact solution $u(t)=cn(2t,\frac{1}{2})$ [8], which is

> $cn := t \to \text{JacobiCN}\left(2\cdot t, \frac{1}{2}\right)$

$$cn := t \mapsto \text{JacobiCN}\left(2\cdot t, \frac{1}{2}\right)$$

> $taylor\_series := series(cn(t), t=0, 18)$

$$taylor\_series := 1-2t^2+\frac{4}{3}t^4-\frac{52}{45}t^6+\frac{46}{45}t^8-\frac{1768}{2025}t^{10}+\frac{50156}{66825}t^{12}-\frac{561224}{868725}t^{14}+\frac{7239772}{13030875}t^{16}+O(t^{18})$$

(24)

> $truncated\_series := convert(taylor\_series, polynom)$

$$truncated\_series := -2t^2+1+\frac{4}{3}t^4-\frac{52}{45}t^6+\frac{46}{45}t^8-\frac{1768}{2025}t^{10}+\frac{50156}{66825}t^{12}-\frac{561224}{868725}t^{14}+\frac{7239772}{13030875}t^{16}$$

Example (2): For $\alpha=\beta=-2$, $u_0=1$ and $u_1=0$, the transform of the initial conditions (14) and the iteration formula (15) reads

$$U(k+2)=\frac{2}{(k+1)(k+2)}\left(U(k)+\sum_{n=0}^{k}\sum_{i=0}^{n}U(i)U(n-i)U(k-n)\right), \quad k=0,1,2,.. \quad (25)$$

$$U(0)=1\ , \qquad U(1)=0 \quad (26)$$

Now, the iteration formula (25) with (26) and making use of Maple-23 Package, enable us to calculate $U(k)$ for any reasonable desire order. Hence, for $k=0,1,2,3,4,5,6,7,8,9,10,11,12$, we found

$U(2)=2$, $U(3)=0$, $U(4)=\dfrac{4}{3}$, $U(5)=0$, $U(6)=\dfrac{52}{45}$,

$U(7)=0$, $U(8)=\dfrac{46}{45}$, $U(9)=0$, $U(10)=\dfrac{1768}{2025}$, $U(11)=0$ (27)

$U(12)=\dfrac{50156}{66825}$, $U(13)=0$, $U(14)=\dfrac{561224}{868725}$,......

This leads to the series solution

$$u(t)=1+2t^2+\frac{4}{3}t^4+\frac{52}{45}t^6+\frac{46}{45}t^8+\frac{1768}{2025}t^{10}+\frac{50156}{66825}t^{12}+\frac{561224}{868725}t^{14}+.....,\tag{28}$$

This is identical to the series solution of the Alvaro and Jairo exact solution $u(t)=nc(2t,\frac{\sqrt{3}}{2})$ [8], which is

> $nc := t \to \text{JacobiNC}\left(2\cdot t, \frac{\sqrt{3}}{2}\right)$

$$nc := t \mapsto \text{JacobiNC}\left(2\cdot t, \frac{\sqrt{3}}{2}\right)$$

> $taylor\_series := series(nc(t), t=0, 18)$

$$taylor\_series := 1+2t^2+\frac{4}{3}t^4+\frac{52}{45}t^6+\frac{46}{45}t^8+\frac{1768}{2025}t^{10}+\frac{50156}{66825}t^{12}+\frac{561224}{868725}t^{14}+\frac{7239772}{13030875}t^{16}+O(t^{18})\tag{29}$$

> $truncated\_series := convert(taylor\_series, polynom)$

$$truncated\_series := 2t^2+1+\frac{4}{3}t^4+\frac{52}{45}t^6+\frac{46}{45}t^8+\frac{1768}{2025}t^{10}+\frac{50156}{66825}t^{12}+\frac{561224}{868725}t^{14}+\frac{7239772}{13030875}t^{16}$$

Example (3): For $\alpha=1$, $\beta=-2$, $u_0=1$ and $u_1=0$, the transform of the initial conditions (14) and the iteration formula (15) reads

$$U(k+2)=\frac{-1}{(k+1)(k+2)}\left(U(k)-2\sum_{n=0}^{k}\sum_{i=0}^{n}U(i)U(n-i)U(k-n)\right),\quad k=0,1,2,..\tag{30}$$

$$U(0)=1,\qquad U(1)=0\tag{31}$$

Now, the iteration formula (30) with (31) and making use of Maple-23 Package, enable us to calculate $U(k)$ for any reasonable desire order. Hence, for $k=0,1,2,3,4,5,6,7,8,9,10,11,12$, we found

$U(2)=\frac{1}{2}$, $U(3)=0$, $U(4)=\frac{5}{24}$, $U(5)=0$, $U(6)=\frac{61}{720}$,

$U(7)=0$, $U(8)=\frac{277}{8064}$, $U(9)=0$, $U(10)=\frac{50521}{3628800}$, (32)

$U(11)=0$, $U(12)=\frac{540553}{95800320}$, $U(13)=0$, $U(14)=\frac{199360981}{87178291200}$

This leads to the series solution

$$u(t)=1+\frac{1}{2}t^2+\frac{5}{24}t^4+\frac{61}{720}t^6+\frac{277}{8064}t^8+\frac{50521}{3628800}t^{10}+\frac{540553}{95800320}t^{12}+\frac{199360981}{87178291200}t^{14}+\ldots \quad (33)$$

This is identical to the series solution of the Alvaro and Jairo exact solution $u(t)=cn(\sqrt{-1}\ \mathrm{t},1)$ [8], which is

> $cn := t \to \mathrm{JacobiCN}(\sqrt{-1}\cdot t, 1)$

$$cn := t \mapsto \mathrm{JacobiCN}(\sqrt{-1}\cdot t, 1)$$

> $taylor\_series := series(cn(t), t=0, 18)$

$$taylor\_series := 1+\frac{1}{2}t^2+\frac{5}{24}t^4+\frac{61}{720}t^6+\frac{277}{8064}t^8+\frac{50521}{3628800}t^{10}+\frac{540553}{95800320}t^{12}+\frac{199360981}{87178291200}t^{14}+\frac{3878302429}{4184557977600}t^{16}+\mathrm{O}(t^{18})$$

(34)

> $truncated\_series := convert(taylor\_series, polynom)$

$$truncated\_series := 1+\frac{1}{2}t^2+\frac{5}{24}t^4+\frac{61}{720}t^6+\frac{277}{8064}t^8+\frac{50521}{3628800}t^{10}+\frac{540553}{95800320}t^{12}+\frac{199360981}{87178291200}t^{14}+\frac{3878302429}{4184557977600}t^{16}$$

Example (4): For $\alpha=3$, $\beta=-1$, $u_0=2$ and $u_1=0$, the transform of the initial conditions (14) and the iteration formula (15) reads

$$U(k+2)=\frac{-1}{(k+1)(k+2)}\left(3U(k)-\sum_{n=0}^{k}\sum_{i=0}^{n}U(i)U(n-i)U(k-n)\right), \quad k=0,1,2,.. \quad (35)$$

$$U(0)=2\ , \quad U(1)=0 \quad (36)$$

Now, the iteration formula (35) with (36) and making use of Maple-23 Package, enable us to calculate $U(k)$ for any reasonable desire order. Hence, for $k=0,1,2,3,4,5,6,7,8,9,10,11,12$, we found

$U(2)=1$, $U(3)=0$, $U(4)=\frac{3}{4}$, $U(5)=0$, $U(6)=\frac{17}{40}$,

$U(7)=0$, $U(8)=\frac{79}{320}$, $U(9)=0$, $U(10)=\frac{1381}{9600}$, (37)

$U(11)=0$, $U(12)=\frac{1071}{12800}$, $U(13)=0$, $U(14)=\frac{16201}{332800}$,....

This leads to the series solution

$$u(t)=2+t^2+\frac{3}{4}t^4+\frac{17}{40}t^6+\frac{79}{320}t^8+\frac{1381}{9600}t^{10}+\frac{1071}{12800}t^{12}+\frac{16201}{332800}t^{14}+\ldots \quad (38)$$

This is identical to the series solution of the Alvaro and Jairo exact solution $u(t)=2cn\left(\sqrt{-1}t,\sqrt{2}\right)$ [8], which is

> $cn := t \to 2\cdot \text{JacobiCN}\left(\sqrt{-1}\cdot t, \sqrt{2}\right)$

$$cn := t \mapsto 2\cdot \text{JacobiCN}\left(\sqrt{-1}\cdot t, \sqrt{2}\right)$$

> $taylor\_series := series(cn(t), t=0, 18)$

$$taylor\_series := 2+t^2+\frac{3}{4}t^4+\frac{17}{40}t^6+\frac{79}{320}t^8+\frac{1381}{9600}t^{10}+\frac{1071}{12800}t^{12}+\frac{16201}{332800}t^{14}+\frac{2262209}{79872000}t^{16}+O(t^{18}) \quad (39)$$

> $truncated\_series := convert(taylor\_series, polynom)$

$$truncated\_series := 2+t^2+\frac{3}{4}t^4+\frac{17}{40}t^6+\frac{79}{320}t^8+\frac{1381}{9600}t^{10}+\frac{1071}{12800}t^{12}+\frac{16201}{332800}t^{14}+\frac{2262209}{79872000}t^{16}$$

Example (5): For $\alpha=2$, $\beta=-1$, $u_0=1$ and $u_1=0$, the transform of the initial conditions (14) and the iteration formula (15) reads

$$U(k+2)=\frac{-1}{(k+1)(k+2)}\left(2U(k)-\sum_{n=0}^{k}\sum_{i=0}^{n}U(i)U(n-i)U(k-n)\right), \quad k=0,1,2,.. \quad (40)$$

$$U(0)=1\ , \qquad U(1)=0 \quad (41)$$

Now, the iteration formula (40) with (41) and making use of Maple-23 Package, enable us to calculate $U(k)$ for any reasonable desire order. Hence, for $k=0,1,2,3,4,5,6,7,8,9,10,11,12$, we found

$$U(2) = -\frac{1}{2}, \quad U(3) = 0, \quad U(4) = -\frac{1}{24}, \quad U(5) = 0, \quad U(6) = \frac{17}{720},$$
$$U(7) = 0, \quad U(8) = \frac{17}{40320}, \quad U(9) = 0, \quad U(10) = -\frac{3889}{3628800}, \tag{42}$$
$$U(11) = 0, \quad U(12) = \frac{24911}{479001600}, \quad U(13) = 0, \quad U(14) = \frac{3846113}{87178291200}, \ldots$$

This leads to the series solution

$$u(t) = 1 - \frac{1}{2}t^2 - \frac{1}{24}t^4 + \frac{17}{720}t^6 + \frac{17}{40320}t^8 - \frac{3889}{3628800}t^{10} + \frac{24911}{479001600}t^{12} + \frac{3846113}{87178291200}t^{14} + \ldots \tag{43}$$

This is identical to the series solution of the Alvaro and Jairo exact solution $u(t) = cn\left(t, \frac{\sqrt{-1}}{\sqrt{2}}\right)$ [8], which is

> $cn := t \rightarrow \text{JacobiCN}\left(t, 1\frac{\sqrt{-1}}{\sqrt{2}}\right)$

$$cn := t \mapsto \text{JacobiCN}\left(t, \frac{\sqrt{-1}}{\sqrt{2}}\right)$$

> $taylor\_series := series(cn(t), t = 0, 18)$

$$taylor\_series := 1 - \frac{1}{2}t^2 - \frac{1}{24}t^4 + \frac{17}{720}t^6 + \frac{17}{40320}t^8 - \frac{3889}{3628800}t^{10} + \frac{24911}{479001600}t^{12} + \frac{3846113}{87178291200}t^{14} - \frac{108848287}{20922789888000}t^{16} + \mathrm{O}(t^{18}) \tag{44}$$

> $truncated\_series := convert(taylor\_series, polynom)$

$$truncated\_series := 1 - \frac{1}{2}t^2 - \frac{1}{24}t^4 + \frac{17}{720}t^6 + \frac{17}{40320}t^8 - \frac{3889}{3628800}t^{10} + \frac{24911}{479001600}t^{12} + \frac{3846113}{87178291200}t^{14} - \frac{108848287}{20922789888000}t^{16}$$

Example (6): For $\alpha = -2$, $\beta = 13$, $u_0 = 10^{-2}$ and $u_1 = 0$, the transform of the initial conditions (14) and the iteration formula (15) reads

$$U(k+2) = \frac{-1}{(k+1)(k+2)}\left(-2U(k) + 13\sum_{n=0}^{k}\sum_{i=0}^{n}U(i)U(n-i)U(k-n)\right), \quad k = 0,1,2,.. \tag{45}$$

$$U(0) = 10^{-2}, \quad U(1) = 0 \tag{46}$$

Now, the iteration formula (40) with (41) and making use of Maple Package, enable us to calculate $U(k)$ for any reasonable desire order. Hence, for $k = 0,1,2,3,4,5,6,7,8,9,10,11,12$, we found

$$U(2) = \frac{19987}{2000000}, \quad U(3) = 0, \quad U(4) = \frac{398960507}{240000000000}, \quad U(5) = 0,$$
$$U(6) = \frac{7870172320681}{72000000000000000}, \quad U(7) = 0, \; U(8) = \frac{19775020308199343}{5760000000000000000000}, \; U(9) = 0,$$
$$U(10) = -\frac{89366610266976019571}{518400000000000000000000000},... \tag{47}$$

This leads to the series solution

$$u(t) = \frac{1}{100} + \frac{19987}{2000000}t^2 + \frac{398960507}{240000000000}t^4 + \frac{7870172320681}{72000000000000000}t^6 + \frac{19775020308199343}{5760000000000000000000}t^8 - \frac{89366610266976019571}{518400000000000000000000000}t^{10} + ... \tag{48}$$

This solution is identical to the series solution of the Alvaro and Jairo exact solution $u(t) = 0.01\, cn\left(\sqrt{-1}\ (1.4142)t, \sqrt{-1}\ (0.018)\right)$ [8], which is

```
> cn := t -> 1/100 · JacobiCN(sqrt(-1) · sqrt(19987/10000) · t, sqrt(-1) · sqrt(13/39974))
```

$$cn := t \mapsto \frac{\mathrm{JacobiCN}\left(\sqrt{-1}\cdot\sqrt{\frac{19987}{10000}}\cdot t, \sqrt{-1}\cdot\sqrt{\frac{13}{39974}}\right)}{100} \tag{49}$$

```
> taylor_series := series(cn(t), t = 0, 12)
```

$$taylor\_series := \frac{1}{100} + \frac{19987}{2000000}t^2 + \frac{398960507}{240000000000}t^4 + \frac{7870172320681}{72000000000000000}t^6 + \frac{19775020308199343}{5760000000000000000000}t^8 - \frac{89366610266976019571}{518400000000000000000000000}t^{10} + O(t^{12})$$

```
> truncated_series := convert(taylor_series, polynom)
```

$$truncated\_series := \frac{1}{100} + \frac{19987}{2000000}t^2 + \frac{398960507}{240000000000}t^4 + \frac{7870172320681}{72000000000000000}t^6 + \frac{19775020308199343}{5760000000000000000000}t^8 - \frac{89366610266976019571}{518400000000000000000000000}t^{10}$$

Conclusion

In this paper we reviewed the differential transform method for solving non-linear ordinary differential equations and introduced a differential transform formula for a cubic non-linear term $f(u) = u^3(t)$. Using this formula for the cubic Duffing equation and with the help of Maple packages, we found that the differential transform method, led to a series solution converges rapidly to the Alvaro and Jairo exact solutions, which obtained in term of the

Jacobian elliptic functions. This study shows that the differential transform method is capable to solve non-linear ordinary differential equations, when the non-linear term is given as a polynomial of degree higher than three.

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